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ISSUSE-1&2

THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

The Association of Mathematics Teachers of India (AMTI) was started in 1965 for the promotion of efforts to improve Mathematics education at all levels. A major aim of the Association is to assist practising teachers of Mathematics in schools in improving their expertise and professional skills. Another important aim is to spot out and foster Mathematical talents in the young. The Association also seeks to disseminate new trends in Mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equipping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centres in schools, in conducting Mathematics laboratory programmes, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds " National Mathematical Talent Search Competition " annually and organizes Orientation Courses, Seminars and Workshops for teachers and courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

. An award for contributions to the Mathematics Teacher relating to History of Mathematics in the context of mathematics education had been instituted by Prof.R.C.Gupta.

"The Mathematics Teacher (India)" (MT) is the official quarterly journal of the Association and is issued twice a year. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in English and Tamil.

The membership of the Association is open to all those interested in Mathematics and Mathematics Education. The membership fee inclusive of subscription for "**The Mathematics Teacher (India)**" and effective from April 1993 is as follows:

Subscription for India*

Category	Individual	Institutional
Life	Rs. 500	Rs. 1000
Annual (Ordinary)	Rs. 50	—
Junior Mathematician - Life	Rs. 250	Rs. 250
Junior Mathematician - Annual	Rs. 25	Rs. 25

The Journal "The Mathematics Teacher" will be supplied free to all members.
Fifty or more subscriptions to Junior Mathematician will entail 20% discount.

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Dr. S.Muralidharan
Editor

EDITORIAL

The summer holidays have just begun. Many of us would have planned for a vacation and looking forward to the travel, seeing new places and meeting new people. The grandeur of nature is always exciting. These travels rejuvenate us.

As mathematics teachers, why don't we also take a "Mathematics vacation"? I suggest we spend some time taking

our minds to meet new concepts in Mathematics, exploring exciting ideas of the great Mathematicians, reading the history of Mathematics and thereby rejuvenate our Math minds. Thanks to the Internet, these things can be done from the comfort of our homes. Resolve to learn either a new concept or explore around something you already know. For example, we know the volume of a cylinder is height times the area of the base and that of cone is $\frac{1}{3}$ times the volume of the cylinder. Is there a solid that is somehow connected to the cylinder that has volume $\frac{1}{2}$ times the volume of the cylinder? How could ancient Greeks find the formula for the volume of a sphere without using Calculus? The possibilities are endless. Explore and enjoy.

This issue contains problems and solutions from the Mathematics competitions held during 2016.

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SCREENING TEST – GAUSS CONTEST
NMTC at PRIMARY LEVEL
V & VI Standards

PART – A

1. The price of an item is decreased by 25%. The percentage increase to be done in the new price to get the original price is

   A. 25% B. 30% C. 43%

   D. $33\frac{1}{3}\%$

   Suppose that the original price of the item is Rs 100. After a 25% reduction it becomes Rs 75. To get the original 100 Rs, if we need to increase by $x\%$, then $75 \times \left(1 + \frac{x}{100}\right) = 100$ and hence $x = 33\frac{1}{3}\%$

2. How many pairs of positive integers are there such that their sum is 528 and their HCF is 33?

   A. 4 B. 6 C. 8 D. 12

   We need to count the number of pairs $(33x, 33y)$ where $\gcd(x, y) = 1$ and $33x + 33y = 528$. Thus $x + y = 16$. Clearly, the pairs (x, y) satisfying the conditions are

   $$(1, 15), (3, 13), (5, 11), (7, 9), (9, 7), (11, 5), (13, 3), (15, 1)$$

   Hence the number of pairs is 8.

3. If one-fifth of two-third of three fourth of a number is 43, then the number is

   A. 256 B. 540 C. 380 D. 430

   Let the number be N. Then we have

   $$\frac{1}{5} \times \frac{2}{3} \times \frac{3}{4} \times N = 43$$

   Hence $N = 430$.

4. The average of 8 numbers is 99. The difference between the two greatest numbers is 18. The average of the remaining 6 numbers is 87. The greater number is

   A. 138 B. 140 C. 144 D. 155

   Let the greatest two numbers be x, y. The total of the eight numbers is $8 \times 99 = 792$. Total of the six numbers other than x, y is $6 \times 87 = 522$. Thus $x + y = 792 - 522 = 270$ and $x - y = 18$. Thus $x = 144, y = 126$ and the greater number is 144.

5. In a rectangle, the length is increased by 40% and breadth decreased by 30%. Then the area is

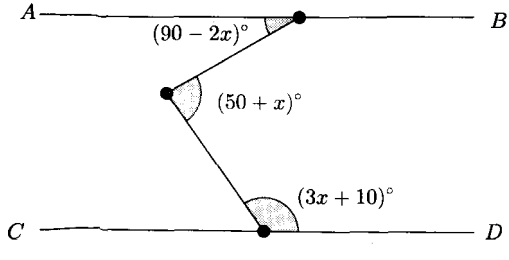
   A. increased by 5% B. decreased by 2%

   C. decreased by 5% D. increased by 2%

   If the original length is L and breadth is B, then the new length and breadth are respectively $1.4L$ and $0.7B$. Thus the new area is $1.4L \times 0.7B = 0.98LB$. Since the original area is LB, the area is decreased by 2%.

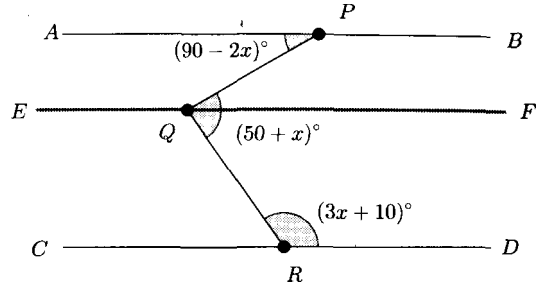
6. In the adjoining figure, lines AB and CD are parallel. What is the value of x in degrees?

   A. 25° B. 35° C. 45°



   D. 55°

   Draw EF parallel to AB. Then $\angle PQF = \angle APQ =$



$(90 - 2x)°$, since these are alternate angles. Also,

$$\angle FQR = 180° - \angle QRD = (180 - 3x - 10)° = (170 - 3x)°$$

since $\angle FQR$ and $\angle QRD$ are supplementary angles. Thus $(50 + x)° = (90 - 2x + 170 - 3x)°$ and $x = 35°$.

7. Find the least among the fractions $\frac{5}{6}, \frac{6}{7}, \frac{7}{8}$

A. $\frac{5}{6}$ B. $\frac{6}{7}$ C. $\frac{7}{8}$ D. $\frac{6}{7}$ and $\frac{7}{8}$

Taking the LCM of the denominators, the given fractions are $\frac{140}{168}, \frac{144}{168}, \frac{147}{168}$ and hence the least of these is $\frac{5}{6}$.

8. By how much is 15% of 23.5 more than 20% of 16.

A. 0.125 B. 0.325 C. 1.5

D. 0.235

15% of 23.5 is $0.15 \times 23.5 = 3.525$ and 20% of 16 is $0.2 \times 16 = 3.2$. Thus it is more by $3.525 - 3.2 = 0.325$.

9. How many times does the digit 1 appear when you write numbers 1 to 399 consecutively?

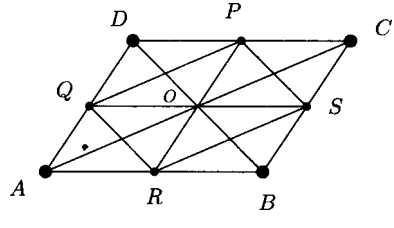
A. 180 B. 175 C. 178 D. 179

Writing the numbers 1 to 9, we write 1 once. In writing

10 to 19, we write 1 eleven times. From 20 to 29, there is only one 1 (in 21), and similarly for 30 to 39, 40 to 49, and so on till 90 to 99. Thus in writing numbers less than 100, we write one for $1+11+1+1+1+1+1+1+1+1=20$ times. Any number from 100 to 199, has 1 in the first place and in the second and third places, the number of ones is same as that in 00 to 99. Hence in writing 100 to 199, we write 1 for $100+20=120$ times. In 200 to 299 and 300 to 399, we write one 20 times each. Thus the total number of ones written is $20+120+20+20=180$ times.

10. The total numbers of parallelograms of different dimensions in the adjoining figures is

A. 14 B. 16 C. 18 D. More

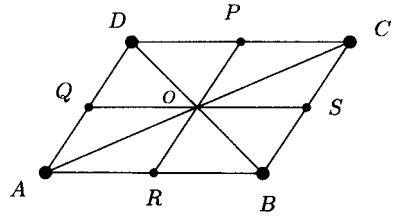


than 18

First consider the parallelogram $ABCD$. We can form nine parallelograms from this:

$$QOPD, AROQ, POSC, SORB$$
$$ARPD, PRBC, DQSC, QABS, ABCD$$

The parallelogran $QRSP$ is similar to $ABCD$ and hence we can find 9 sub parallelograms from this. Since



these two sets have no parallelograms in common, the total number of parallelograms is $9 + 9 = 18$.

PART – B

11. Consider the sequence $\frac{3}{5}, \frac{6}{7}, 1, 1\frac{1}{11}, \ldots$. The 2016 th term of this sequence is $\frac{p}{q}$ where p, q are integers having no common factors, the value of $q - p$ is ———
The given fractions can be written as

$$\frac{3}{5}, \frac{6}{7} = \frac{3+3}{5+2}, 1 = \frac{6+3}{7+2}, 1\frac{1}{11} = \frac{9+3}{9+2}, \cdots$$

Hence for the n th fraction, the numerator is $3n$ and the denominator is $5 + 2(n - 1)$. Thus the 2016th term is

$$\frac{2016 \times 3}{5 + 2 \times 2015} = \frac{6048}{4035} = \frac{2016}{1345}$$

Since 2016 and 1345 have no common divisors, it follows that $q - p = 1345 - 2016 = -641$.

12. The number of 3 digit numbers that contain 7 as at least one of the digits is ———
The total number of three digit numbers is 900. Of these let us count those that do not contain 7. The first digit can be any digit $\neq 0, 7$, second and third can be any digit

$\neq 7$. Thus there are $8 \times 9 \times 9 = 648$ three digit numbers that do not contain 7. It follows that the number of three digit numbers that contain 7 as at least one of the digits is $900 - 648 = 252$.

13. Mahadevan conducted a problem solving session for a group of 18 primary class students. Seeing the graded performance, he distributed packets of biscuits to all the students.
The first student got 1 packet plus $\frac{9}{19}$ of 1 packet.

The second student got 2 packets plus $\frac{9}{19}$ of 2 packets.

The third student got 3 packets plus $\frac{9}{19}$ of 3 packets and so an.

The total number of packets of biscuits distributed by Mahadeven is ———
The ith student got $i + \frac{9}{19} \times i = i\left(1 + \frac{9}{19}\right) = \frac{28i}{19}$ packets. Thus the total number of biscuit packets is

$$\frac{28}{19}(1 + 2 + \cdots + 18) = \frac{28}{19} \times \left(\frac{18 \times 19}{2}\right) = 9 \times 28 = 252$$

14. Using the digits of the number 2016, two digit numbers of different digits are formed. The sum of all these numbers is ———
The two digit numbers that can be formed using the digits of 2016 are

$$10, 20, 60, 12, 21, 16, 61, 26, 62$$

and hence their sum is 288.

15. The least multiple of 7, that leaves a remainder 4 when divided by 6, 9, 15 and 18 is ———

The least common multiple of $6, 9, 15$ is 90. Any number that leaves a remainder 4 when divided by 6, 9, 15 also leave the same remainder 4 when divided by their least common multiple. Thus we need to find the smallest multiple of 7 among $94, 184, 274, 364, \ldots$. Clearly among these 364 is the smallest number that is also a multiple of 7.

16. The number of revolutions that a wheel of diameter $\frac{7}{11}$ meter will make in going 8 kilometers on a level road is ———

When a wheel of diameter d makes one revolution, it travels a distance of πd. Thus we need to find n such that $\pi \times n \times d = 8000$. Taking $\pi = \frac{22}{7}$, we get $\frac{22}{7} \times \frac{7}{11} \times n = 8000$ and hence $n = 4000$.

17. The radius of a circle is increased so that its circumference increases by 5%. The area of the circle will increase (in %) by ———

If the original radius is r and the new radius is R, then $\frac{2\pi R}{2\pi r} = 1.05$ and hence $\frac{R}{r} = 1.05$. The ratio of their areas is

$$\frac{\pi R^2}{\pi r^2} = \left(\frac{R}{r}\right)^2 = 1.05^2 = 1.1025$$

Thus the area increases by $10.25\,\%$

18. The sum of seven numbers is 235. The average of the first three is 23 and that of the last three is 42. The fourth number is ———

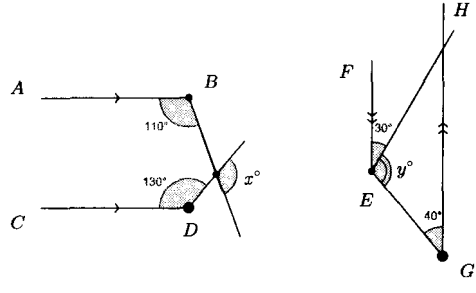
The sum of the first three numbers is $3 \times 23 = 69$ and the sum of the last three is $3 \times 42 = 126$. Thus the fourth number is $235 - 69 - 126 = 40$

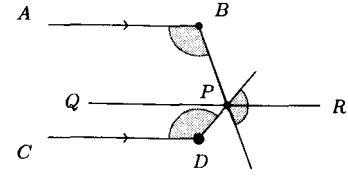
19. The number of $\frac{1}{6}$ that are in $116\frac{2}{3}$ is ———

Since $116\frac{2}{3} = \frac{350}{3}$, the number of $\frac{1}{6}$ in this number is

$$\frac{\frac{350}{3}}{\frac{1}{6}} = \frac{350}{3} \times \frac{6}{1} = 700$$

20. In the figure below, AB is parallel to CD and EF is parallel to GH. The value of $x° - y°$ is ———





Since $\angle FEH$ and $\angle EHG$ are alternate angles, they are equal. Thus from triangle EGH, we have $y = 180° - 40° - 30° = 110°$.

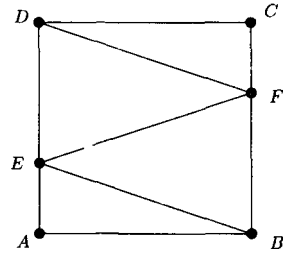
Draw a parallel line QR through P as shown in the figure. Angles QPD and PDC are supplementary. Thus $\angle QPD = 180° - 130° = 50°$. Similarly, $\angle BPQ = 70°$. Thus $x = \angle BPD = 50° + 70° = 120°$. Thus $x - y = 10°$

SCREENING TEST – KAPREKAR CONTEST
NMTC at SUB JUNIOR LEVEL
VII & VIII STANDARDS

PART – A

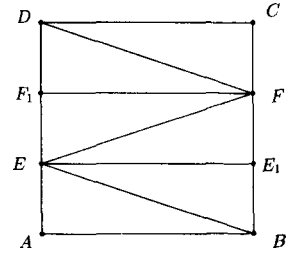
1. On the square $ABCD$, point E lies on the side AD and F lies on BC so that $BE = EF = FD = 30\,\text{cm}$. The area of the square (in square cms) is

A. 300 B. 900 C. 810



D. None of these

Draw the perpendiculars from E, F to the opposite



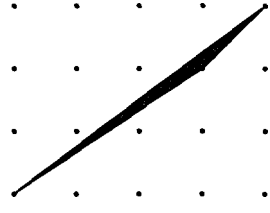
sides. If these meet the sides at E_1, F_1, clearly, $AE =$

A. 10 B. 30 C. $10\sqrt{2}$
D. $10\sqrt{3}$

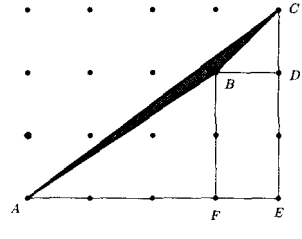
If the side of the cube is a, the surface area is $6a^2$. Hence we have $6a^2 = 600$ and thus $a = 10$. The length of the diagonal is $\sqrt{a^2 + a^2 + a^2} = 10\sqrt{3}$

9. In the figure below the distance between any two horizontal or vertical dots is one unit. The area of the triangle shown is

A. $\frac{1}{2}$ B. $\frac{1}{3}$ C. $\frac{1}{4}$ D. $\frac{1}{6}$



The area of the triangle ABC equals area of triangle



AEC – area of triangle ABF – area of triangle BCD – area of rectangle $BDEF$ and hence equal to

$$\frac{1}{2} \times 4 \times 3 - \frac{1}{2} \times 3 \times 2 - \frac{1}{2} \times 1 \times 1 - 2 \times 1 = \frac{1}{2}$$

10. The number of natural numbers less than 400 that are not divisible by 17 or 23 is

A. 290 B. 320 C. 360 D. 370

The number of natural numbers less than 400 and divisible by 17 is $\left[\frac{399}{17}\right] = 23$ and the number of natural numbers less than 400 and divisible by 23 is $\left[\frac{399}{23}\right] = 17$ (Here for any real number x, $[x]$ denotes the integral part of x). Since $17 \times 23 = 391$ it follows that the number of natural numbers that are divisible by 17 or 23 is $23+17-1 = 39$ and thus the number of numbers that are neither divisible by 17 nor by 23 is $399 - 39 = 360$.

11. If a is a % of b and b is b% of c, where a is a positive real number, then the value of c is

A. 120 B. 200 C. 150 D. 100

We have $a = \frac{ab}{100}$ and $b = \frac{bc}{100}$ and hence $c = 100$.

12. The sum of the reciprocals of all the positive integers that divide 24 is

A. $\frac{7}{2}$ B. $\frac{1}{2}$ C. $\frac{5}{2}$ D. $\frac{3}{2}$

The divisors of 24 are $1, 2, 3, 4, 6, 8, 12, 24$. Hence the sum of their reciprocals is

$$\frac{1}{1}+\frac{1}{24}+\frac{1}{2}+\frac{1}{12}+\frac{1}{3}+\frac{1}{8}+\frac{1}{4}+\frac{1}{6} = \frac{25+14+11+10}{24} = \frac{5}{2}$$

13. N is a positive integer and p, q are primes. If $N = pq$ and $\frac{1}{N}+\frac{1}{p} = \frac{1}{q}$, the value of N is

A. 6 B. 7 C. 8 D. 9

We have $\frac{1}{pq} = \frac{1}{q} - \frac{1}{p} = \frac{p-q}{pq}$. Thus $p - q = 1$. Since

p, q are primes the only possibility is $p = 3, q = 2$. Thus $N = 6$.

14. The number of two digit numbers that leave a remainder 1 when divided by 4 is

A. 20 B. 21 C. 22 D. 23

The smallest two digit number that leaves a remainder 1 when divided by 4 is 13 and the largest such two digit number is 97. Hence the list of such numbers is $13, 17, 21, \ldots, 97$. The number of such numbers is $1 + \frac{97-13}{4} = 22$.

15. If $(a+b)^2 + (b+c)^2 + (c+d)^2 = 4(ab + bc + cd)$, then

A. $a + b$ or $b + c$ or $c + d$ must be zero

B. Two of a, b, c, d are zero and other two non zero

C. $a = b = c = d$

D. None of these

Bringing all the terms to the left hand side, we obtain

$$(a-b)^2 + (b-c)^2 + (c-d)^2 = 0$$

and hence $a = b = c = d$.

PART – B

16. A race horse eats $(3a + 2b)$ bags of oats every week. The number of weeks in which it can eat $(12a^2 - 7ab - 10b^2)$ bags of oats is ———

The number of weeks is given by

$$\frac{12a^2 - 7ab - 10b^2}{3a + 2b} = 4a - 5b$$

17. Choose 4 digits a, b, c, d from $\{2, 0, 1, 6\}$ and form the number $(10a + b)^{10c+d}$. For example, if $a = 2, b = 0, c =$

$1, d = 6$, we will get 20^{16}. For all such choices of a, b, c, d the number of distinct numbers that will be formed is —

We can choose a in 4 ways and once a is chosen, b can be chosen in 3 ways. Now there are 2 choices for c and one choice for d. Hence there are 24 possibilities. But not all of these give distinct numbers. When $a = 0, b = 1$, both the choices $c = 2, d = 6$ and $c = 6, d = 2$ give the number 1. For the other choices we obtain distinct numbers and hence the number of distinct integers we obtain is 23.

18. A fraction F becomes $\frac{1}{2}$ when its denominator is increased by 4 and becomes $\frac{1}{3}$ when its numerator is decreased by 5. Then F equals ———
Let $F = \frac{p}{q}$. We have $\frac{p}{q+4} = \frac{1}{2}$ and $\frac{p-5}{q} = \frac{1}{3}$. Hence $2p = q + 4$ and $3p - 15 = q$. Solving for p, q we get $p = 11$ and $q = 18$. The fraction is $\frac{11}{18}$.

19. The average of 5 consecutive positive integers starting with m is n. Then the average of 5 consecutive integers starting with n is (in terms of m) is ———
We have $m + (m + 1) + (m + 2) + (m + 3) + (m + 4) = 5m + 10$ and hence $m + 2 = n$. Thus the average of 5 consecutive integers starting with n is $\frac{5n + 10}{5} = n+2 = m + 4$

20. Two boys came to Mahadevan and asked his age. Mahadevan told, "Delete all the vowels and repeated letters from my name. Find the numerical value of the remaining letters (for example, D has value 4, G has 7 etc). Add all of them. Find the number got by interchanging its digits. Add both the numbers. That

is my age". One boy ran away. The other boy calculated correctly. The age of Mahadevan is ———

The vowels are A and E and there are no repeated non vowels. Hence after removal, we obtain $MHDVN$ and adding the numerical values of these, we obtain $13+8+4+22+14=61$. Hence the age is $61+16=77$.

21. If

$$A=\frac{2^4+2^4}{2^{-4}+2^{-4}},\quad B=\frac{3^2+3^2}{3^{-2}+3^{-2}},\quad C=\frac{4^2+4^2}{4^{-2}+4^{-2}}$$

the integral part of $\dfrac{A+C}{B}$ is ———

$A=\dfrac{2^5}{2^{-3}}=2^8, B=\dfrac{2\cdot 3^2}{2\cdot 3^{-2}}=3^4$ and $C=\dfrac{2\cdot 4^2}{2\cdot 4^{-2}}=4^4$

Thus $\dfrac{A+C}{B}=\dfrac{2^8+4^4}{3^4}=\dfrac{512}{81}$ and the integral part is 6.

22. A six-digit number is formed using the digits 1, 1, 2, 2, 3, 3. The number of 6-digit numbers in which the 1s are separated by one digit, 2s are separated by two digits and 3s are separated by 3 digits is ———

The number 3 can be only in positions 1 and 5 or 2 and 6. Hence the numbers are 312132 and 231213.

23. In the addition shown below, P, I, U are digits. The value of U is ———

$$PI+PI+PI+PI=UP$$

We have $4(10P+I)=10U+P$ and hence $39P=10U-4I$. This implies that P is even. If $P\geq 4$, then $4\times PI$ will have more than 2 digits. Thus $P=2$ and $78=10U-4I$. The only possibility is $U=9$ and $I=3$.

24. There are four cows, eight hen, a fish, a crow, a girl and a boy in a garden. Outside the garden there is one dog, a peacock and some cats. The number of legs of all of them inside the garden is equal that outside the garden. The number of cats is ———
The number of legs inside the garden is

$$4 \times 4 + 8 \times 2 + 1 \times 0 + 1 \times 2 + 1 \times 2 + 1 \times 2 = 38$$

If the number of cats is c, then the number of legs outside the garden is $1 \times 4 + 1 \times 2 + c \times 4 = 4c + 6$. Hence the number of cats is 8.

25. Two sides of a triangle are 8 cm and 5 cm. The length of the third side in cms is also an integer. The number of such triangles is ———
If the length of the third side is c, then we must have $c + 5 > 8$ and $8 + 5 > c$. Thus $3 < c < 13$. Hence $c = 4, 5, 6, 7, 8, 9, 10, 11, 12$ and there are 9 such triangles.

26. If $a^2 - a - 10 = 0$, then $(a + 1)(a + 2)(a - 4)$ is ———

$$\begin{aligned}(a + 1)(a + 2)(a - 4) &= a^3 - a^2 - 10a - 8 \\ &= a(a^2 - a - 10) - 8 \\ &= -8\end{aligned}$$

27. There are 5 points on the circumference of a circle. The number of chords which can be drawn joining them is —
If the points are A, B, C, D, E, then the chords are $AB, AC, AD, AE, BC, BD, BE, CD, CE, DE$ and hence there are 10 chords.

28. Each side of an equilateral triangle is 3 cm longer than each side of a square. The total perimeter of the square

and the triangle is 51 cm. Then the side of the triangle in cms is ———

Let a be the side of the square. Then the side of the equilateral triangle is $a + 3$. The perimeter of the triangle is $3a + 9$ and that of the square is $4a$. We have $3a + 9 + 4a = 51$ and hence $a = 6$. Thus the side of the triangle in cms is 9.

29. The largest three digit number that is a multiple of 3 and 5 is ———

The largest three digit number is 999 and when this number is divided by 15, we get a remainder 9. Hence the largest three digit number that is a multiple of 3 and 5 is 990.

30. Consider the sequence $0, 6, 24, 60, 120, \ldots$. The 6th term of this sequence is ———

Observe the following table:

0		6		24		60		120
	6		18		36		60	
		12		18		24		
			6		6			

The second row is obtained by subtracting consecutive elements in the first row and similarly third row is obtained from the second row. In the fourth row, the numbers are all same and hence we can extend by inserting a 6 in that row and computing backwards. We will obtain 30 as the next term in the third row as 30, the next term in the second row as 90 and the next term in the first row as 210. Thus the sixth term of the sequence is 210.

SCREENING TEST – BHASKARA CONTEST
NMTC at JUNIOR LEVEL
IX & X Standards

PART – A

1. The sum of the values x, y that satisfy the equations

$$(x+y)2^{y-x} = 1, \quad (x+y)^{x-y} = 2$$

simultaneously is

A. 2 B. $\frac{3}{2}$ C. $\frac{5}{2}$ D. $\frac{7}{2}$

From the second equation, we get $x + y = 2^{1/(x-y)}$. Substituting this in the first equation, we get

$$2^{\frac{1}{(x-y)}} 2^{y-x} = 2^{\frac{1}{x-y}+(y-x)} = 1$$

and hence $\frac{1}{x-y} + y - x = 0$. This gives $(x-y)^2 = 1$ and thus $x - y = \pm 1$. If $x - y = 1$, then from the second equation, we get $x + y = 2$ and $(x, y) = \left(\frac{3}{2}, \frac{1}{2}\right)$ is a solution. If $x - y = -1$, then $x + y = \frac{1}{2}$ and $(x, y) = \left(-\frac{1}{4}, \frac{3}{4}\right)$ is a solution. The sum of all values of x, y that satisfy the given equations is $\frac{3}{2} + \frac{1}{2} - \frac{1}{4} + \frac{3}{4} = \frac{5}{2}$

2. If

$$\left(2 - \frac{a}{4} - \frac{4}{a}\right) \times \left\{(a-4)\sqrt[3]{(a-4)^{-3}} - \frac{(a^2-16)^{-\frac{1}{2}}(a-4)^{-\frac{1}{2}}}{(a+4)^{-\frac{3}{2}}}\right\} \times \left(\frac{a+4}{a-4}\right) = 2016$$

the value of a is

A. $\frac{4}{1007}$ B. $\frac{3}{2016}$ C. $\frac{4}{2017}$

D. none of these

$$(a-4)\sqrt[3]{(a-4)^{-3}} = (a-4)(a-4)^{-1} = 1$$

$$\frac{(a^2-16)^{-\frac{1}{2}}(a-4)^{-\frac{1}{2}}}{(a+4)^{-\frac{3}{2}}} = \frac{(a-4)^{-1/2}(a+4)^{-1/2}(a-4)^{-1/2}}{(a+4)^{-3/2}}$$

$$= \frac{a+4}{a-4}$$

$$2 - \frac{a}{4} - \frac{4}{a} = \frac{8a - a^2 - 16}{4a} = -\frac{(a-4)^2}{4a}$$

Hence the given expression equals

$$-\frac{(a-4)^2}{4a} \times \left(1 - \frac{a+4}{a-4}\right) \times \frac{a+4}{a-4}$$

$$= -\frac{(a-4)^2}{4a} \times \left(\frac{-8}{a-4}\right) \times \frac{a+4}{a-4}$$

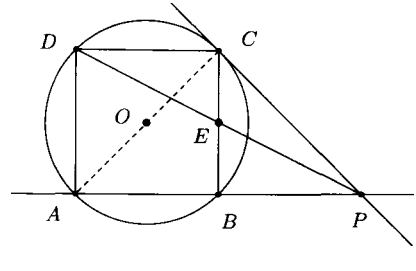
$$= \frac{2(a+4)}{a}$$

Thus $\frac{2(a+4)}{a} = 2016$ and it follows that $a = \frac{4}{1007}$

3. $ABCD$ is a square inscribed in a circle of radius 1 unit. The tangent to the circle at C meets AB produced at P. The length of PD is

A. 2 B. 3 C. $\sqrt{13}$ D. $\sqrt{10}$

Let E be the point of intersection of PD and BC.



Clearly, triangles CDE and BPE are similar. Also, the

line CA passes through the center O of the circle and $\angle PCE = \angle CAB = 45°$. Thus $CB = BP = CD$. Thus the triangles CDE and BPE are congruent. Also, since the radius of the circle is 1, it follows that the length of AB is $\sqrt{2}$ and $BE = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$. Thus

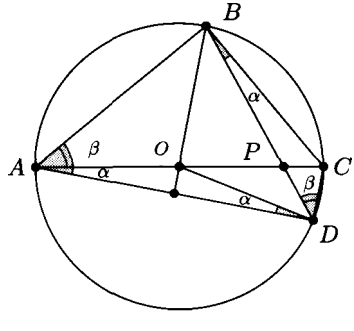
$$PE^2 = BP^2 + BE^2 = 2 + \frac{1}{2} = \frac{5}{2}$$

Thus $PE = \sqrt{\frac{5}{2}}$ and hence $PD = 2PE = 2\sqrt{\frac{5}{2}} = \sqrt{10}$

4. Quadrilateral $ABCD$ is inscribed in a circle with radius 1 unit. AC is the diameter of the circle and $BD = AB$. The diagonals cut at P. If $PC = \frac{2}{5}$ then the length of CD is equal to

A. $\frac{2}{3}$ B. $\frac{2}{7}$ C. $\frac{1}{8}$ D. $\frac{3}{4}$

Let O be the center of the circle. Since $AB = BD$,



$\angle BAD = \angle BDA$. Also, $\angle OAD = \angle ODA$. Now,

$$\angle CDP = \angle CAB = \angle BAD - \angle OAD$$
$$\angle ODP = \angle BDA - \angle ODA$$
$$= \angle BAD - \angle OAD = \angle CDP$$

Thus DP bisects $\angle CDO$ and

$$\frac{CD}{OD} = \frac{PC}{PO} = \frac{2/5}{3/5} = \frac{2}{3}$$

5. The number of natural numbers n for which the expression $\dfrac{23n^2 + 18n + 4}{n}$ is also a natural number is

A. 3 B. 2 C. 1 D. 0

Since $\frac{23n^2+18n+4}{n} = 23n + 18 + \frac{4}{n}$, the expression is a natural number only when n divides 4. The possible natural numbers n that divide 4 are $1, 2, 4$ and hence there are three such natural numbers.

6. The cost price of 16 oranges is equal to the selling price of 12 oranges. Then there is a

A. 40% profit B. 20% loss C. $33\frac{1}{3}$ % profit
D. $23\frac{1}{3}$ % profit

Suppose that the cost price of an orange is C and S its selling price. We have $16C = 12S$ and hence $\frac{S}{C} = \frac{4}{3}$. The profit percentage is $100 \times \frac{S-C}{C} = 100 \times \frac{1}{3} = 33\frac{1}{3}\%$

7. The number of positive integer pairs (a, b) such that $ab - 24 = 2b$ is

A. 6 B. 7 C. 8 D. 9

From $ab - 24 = 2b$, we get $(a - 2)b = 24$. The possible integer values for b are $1, 2, 4, 8, 3, 6, 12, 24$. Since $(a - 2)b = 24$, from the value of b, the value of a is uniquely determined and hence the number of pairs is 8.

8. $A = (2+1)(2^2+1)(2^4+1)\cdots(2^{2016}+1)$. The value of $(A+1)^{1/2016}$ is

   A. 4 B. 2016 C. 2^{4032} D. 2

$$\begin{aligned} A &= A(2-1) \\ &= (2-1)(2+1)(2^2+1)(2^4+1)\cdots(2^{2016}+1) \\ &= (2^2-1)(2^2+1)(2^4+1)\cdots(2^{2016}+1) \\ &= (2^4-1)(2^4+1)\cdots(2^{2016}+1) \\ &= (2^{2016}-1)(2^{2016}+1) \\ &= 2^{4032}-1 \end{aligned}$$

   Thus $(A+1)^{1/2016} = \left(2^{4032}\right)^{1/2016} = 4$

9. The sum of two numbers a, b where $a < b$ is 1215 and their H.C.F is 81. The number of pairs of such pairs (a, b) is

   A. 1 B. 2 C. 3 D. 4

   Since the H.C.F of a, b is 81, we can write $a = 81m$, $b = 81n$ where m, n are coprime. Since $a+b = 1215$, we have $m+n = 15$. Since $m > n$, the possible values for (m, n) are $(14,1), (13,2), (11,4), (8,7)$. Thus there four such pairs.

10. The first Republic Day of India was celebrated on 26th January 1950. What was the day of the week on that date?

    A. Tuesday B. Wednesday C. Thursday D. Friday

    27th August, 2016 (the date of this Screening test) is a Saturday. Let us count the number of days from 26th January 1950 to 27th August, 2016. Between 1951 and 2015, there are 16 leap years and 49 non leap years. Between 26th January and 31st December of 1950, there are 339 days and between January 1st, 2016 and 27th

August 2016, there are 240 days. Hence the total number of days between 26th January 1950 and 27th August 2016 is

$$16 \times 366 + 49 \times 365 + 339 + 240 = 24320$$

Since $24320 \mod 7 = 2$, it follows that 26th January 1050 is a Thursday.

Remark There are various methods to calculate the day of the week for any particular date in the past or future. These methods ultimately rely on algorithms to determine the day of the week for any given date. Gauss described an algorithm to calculate the day of the week of 1st January of any year. He numbered the weekdays from 0 to 6 starting with Sunday. If the year number is Y, then the day of the week of 1st January of the year Y is given by

$$R(1 + 5R(Y-1,4) + 4R(Y-1,100) + 6R(Y-1,400), 7)$$

where $R(a,b)$ is the remainder obtained when a is divided by b.

In the current case, $Y = 1950$ and hence we get the day of the week for 1st January 1950 as Sunday. Hence 26th January 1950 is a Thursday.

11. The 12 numbers $a_1, a_2, \ldots, a_{12}$ are in arithmetical progression. The sum of all these numbers is 354. Let $P = a_2 + a_4 + \cdots + a_{12}$ and $Q = a_1 + a_3 + \cdots + a_{11}$. If the ratio $P : Q$ is $32 : 27$, the common difference of the progression is

A. 2 B. 3 C. 4 D. 5

Let the progression be $a, a + d, \ldots, a + 11d$.

$$a_1 + a_2 + \cdots + a_{12} = 6(2a + 11d) = 354$$
$$P = a_2 + a_4 + \cdots + a_{12} = 3(2a + 12d)$$
$$Q = a_1 + a_3 + \cdots + a_{11} = 3(2a + 10d)$$

Hence we have

$$2a + 11d = 59$$
$$2a + 12d : 2a + 10d = 32 : 27 \Rightarrow 5a = 2d$$

Solving for d, we obtain $d = 5$.

12. A shopkeeper marks the prices of his goods at 20% higher than the original price. There is an increase in demand of the goods, and he further increases the price by 20%. The total profit % is

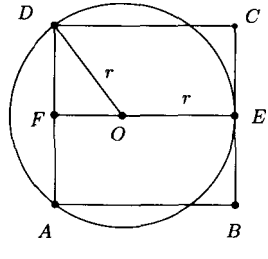
A. 40 B. 38 C. 42 D. 44

If the original price is Rs 100, the 20% mark up price is Rs 120. Another 20% increase is an increase of $120 \times \frac{20}{100} = 24$ rupees and hence the new price is Rs 144 and the profit is 44 %

13. A circle passes through the vertices A and D and touches the side BC of a square $ABCD$. The side of the square is 2 cm. The radius of the circle (in cm) is

A. $\frac{5}{4}$ B. $\frac{4}{5}$ C. 1 D. $\frac{5}{2}$

Since the circle passes through A, D, its center is on the



perpendicular bisector of AD and the point of contact with BC must also lie on this bisector. Let r be the

radius of the circle. Then from the right angled triangle OFD, we have $r^2 - (2 - r)^2 = 1$ and hence $r = \frac{5}{4}$.

14. There are four balls – one green, one red, one blue and one yellow and there are four boxes – one green, one red, one blue and one yellow. A child playing with the balls decides to put the balls in the boxes, one ball in each box. The number of ways in which the child can put the balls in the boxes such that no ball is in a box of its own color is

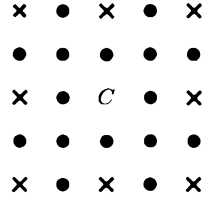
A. 12 B. 9 C. 24 D. 6

The different possibilities are shown below:

$$(R,G,Y,B),(R,Y,G,B),(R,B,Y,G)$$
$$(Y,G,R,B),(Y,B,G,R),(Y,B,R,G)$$
$$(B,G,Y,R),(B,Y,G,R),(B,Y,R,G)$$

where (X,Y,Z,T) means ball colored X is put in Green box, Y is put in Red box, Z is put in Blue box and T is put in Yellow box. Thus there are 9 arrangements.

15. The 5×5 array of dots represents trees in an orchard. If you were standing at the central spot marked C, you

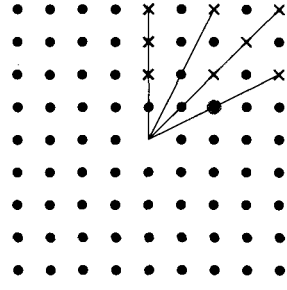


would not be able to see 8 of the 24 trees (shown as X). If you were standing at the center of a 9×9 array of

trees, how many of the 80 trees would be hidden?

A. 40 B. 32 C. 36 D. 44

Consider the trees in the top right quadrant. As shown,



we would not be able to see 8 of these trees (marked X). Hence the total number of trees hidden is 32.

PART – B

16. a and b are positive integers such that $a^2 + 2b = b^2 + 2a + 5$. The value of b is ———

From the given equation, we get

$$(a^2 - 2a + 1) = (b^2 - 2b + 1) + 5$$
$$(a-1)^2 - (b-1)^2 = 5$$
$$(a + b - 2)(a - b) = 5$$

Since a, b are positive integers, we have $a + b - 2 = 5, a - b = 1$ and hence $b = 3$.

17. After full simplification, the value of the product

$$\left(\sqrt{2+\sqrt{3}}\right)\left(\sqrt{2+\sqrt{2+\sqrt{3}}}\right) \times \left(\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}\right)\left(\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}\right)$$

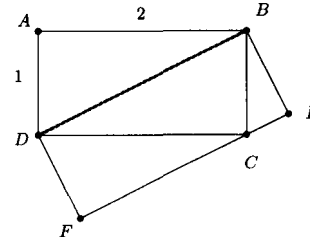
is ———

$$\left(\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}\right)\left(\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}\right)$$
$$=\sqrt{4-\left(2+\sqrt{2+\sqrt{3}}\right)}$$
$$=\sqrt{2-\sqrt{2+\sqrt{3}}}$$

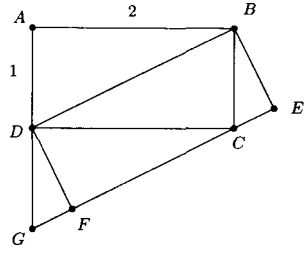
$$\left(\sqrt{2+\sqrt{2+\sqrt{3}}}\right)\left(\sqrt{2-\sqrt{2+\sqrt{3}}}\right)$$
$$=\sqrt{4-(2+\sqrt{3})}$$
$$=\sqrt{2-\sqrt{3}}$$

$$\left(\sqrt{2+\sqrt{3}}\right)\left(\sqrt{2-\sqrt{3}}\right)$$
$$=\sqrt{4-3}=1$$

18. $ABCD$ is a rectangle with $AD = 1$ and $AB = 2$. $DFEB$ is also a rectangle. The area of $DFEB$ is ——



Extend AD and let it meet EF extended at G. Clearly triangles BCE and DGF are congruent. Hence area of rectangle $DFEB$ is equal to the area of the parallelogram $DGCB$. But area of $DGCB$ is sum



of areas of triangles DCB and GDC. Each of these triangles has area equal to half the area of the rectangle $ABCD$. Thus area of the parallelogram $DGCB$ is 2. Thus the area of $DFEB$ also is 2.

19. The two digit number whose units digit exceeds the tens digit by 2 and such that the product of the number and the sum of its digits is 144 is ———
Let the number be $10a + (a + 2)$. We have

$$(2a + 2)(10a + a + 2) = 144 \Rightarrow (a + 1)(11a + 2) = 72$$
$$\Rightarrow 11a^2 + 13a - 70 = 0 \Rightarrow (a - 2)(11a + 35) = 0$$

Thus $a = 2$ and the number is 24.

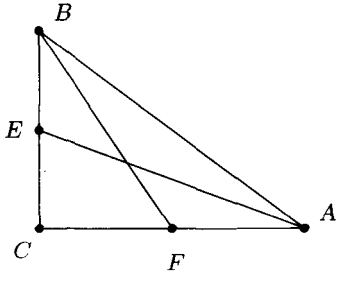
20. If $x = \frac{p}{q}$ where p, q are integers having no common divisors other than 1, satisfies

$$\sqrt{x + \sqrt{x}} - \sqrt{x - \sqrt{x}} = \frac{3}{2}\sqrt{\frac{x}{x + \sqrt{x}}}$$

then x is ———

$$
\begin{aligned}
& x+\sqrt{x}-\sqrt{x^2-x}=\frac{3}{2}\sqrt{x} \\
\Rightarrow\ & \sqrt{x}+1-\sqrt{x-1}=\frac{3}{2} \\
\Rightarrow\ & \sqrt{x}-\sqrt{x-1}=\frac{1}{2} \\
\Rightarrow\ & x+(x-1)-2\sqrt{x(x-1)}=\frac{1}{4} \\
\Rightarrow\ & 2x-\frac{5}{4}=2\sqrt{x(x-1)} \\
\Rightarrow\ & 4x^2+\frac{25}{16}-5x=4x(x-1) \\
\Rightarrow\ & x=\frac{25}{16}
\end{aligned}
$$

21. AE and BF are medians drawn to the legs of a right angled triangle ABC. The numerical value of $\dfrac{AE^2+BF^2}{AB^2}$ is ———



$$AE^2 = AC^2 + EC^2 = AC^2 + \left(\frac{1}{2}BC\right)^2$$

$$= AC^2 + \frac{1}{4}BC^2$$

$$BF^2 = BC^2 + CF^2 = BC^2 + \left(\frac{1}{2}AC\right)^2$$

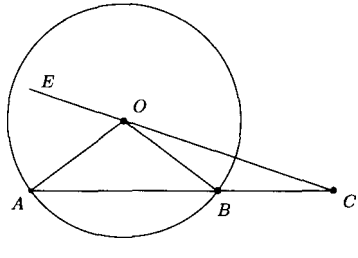
$$= BC^2 + \frac{1}{4}AC^2$$

$$AE^2 + BF^2 = \frac{5}{4}(AC^2 + BC^2) = \frac{5}{4}AB^2$$

Hence $\dfrac{AE^2 + BF^2}{AB^2} = \dfrac{5}{4}$.

22. AB is a chord of a circle with center O. AB is produced to C such that $BC = OA$. CO is produced to E. The value of $\dfrac{\angle AOE}{\angle ACE}$ is ———

Let $\angle ACE = \theta$. Since $BC = OA = OB$, it follows that



$\angle BOC = \angle BCO = \theta$. Thus $\angle OBA = 2\theta = \angle OAB$. Hence $\angle AOB = 180° - 4\theta$. Thus

$$\angle AOE = 180° - \angle COB - \angle AOB = 3\theta = 3\angle ACE$$

The ratio $\dfrac{\angle AOE}{\angle ACE} = 3$.

23. The number of two digit numbers that are less than the sum of the squares of their digits by 11 and exceed twice the product of their digits by 5 is ———

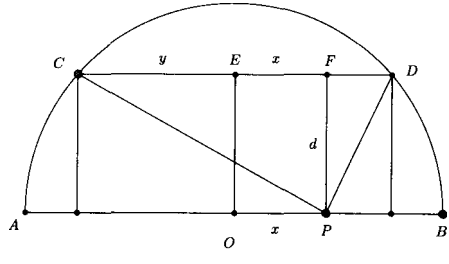
Let $10a + b$ be one such number. Then $10a + b = a^2 + b^2 - 11$ and $10a + b = 2ab + 5$. Hence $a^2 + b^2 - 11 = 2ab + 5 \Rightarrow (a - b)^2 = 16$. Thus $a - b = \pm 4$. The possibilities for (a, b) are

$$(1,5), (2,6), (3,7), (4,8), (5,9),$$
$$(4,0), (5,1), (6,2), (7,3), (8,4), (9,5)$$

Of these only when $(a, b) = (1,5)$ and $(9,5)$, both the given conditions are satisfied. Thus there are two such numbers.

24. AB is a diameter of circle and CD is a parallel chord. P is any point in AB. The numerical value of $\dfrac{PC^2 + PD^2}{PA^2 + PB^2}$ is ———

Let O be the center of the circle. Let $OP = x$ and



d distance of the chord CD from the parallel diameter AB. Let E be the mid point of the chord CD and $EC = y$. We have

$$\begin{aligned}(PC^2 - d^2) + (PD^2 - d^2) &= CF^2 + FD^2 \\ &= (y + x)^2 + (y - x)^2 = 2(x^2 + y^2)\end{aligned}$$

Thus

$$PC^2 + PD^2 = 2(x^2 + y^2) + 2d^2 = 2x^2 + 2(y^2 + d^2)$$

But $y^2 + d^2 = CE^2 + OE^2 = OC^2 = r^2$. Thus $PC^2 + PD^2 = 2(x^2 + r^2)$. Again,

$$PA^2 + PB^2 = (r + x)^2 + (r - x)^2 = 2(x^2 + r^2)$$

Thus $PC^2 + PD^2 = PA^2 + PB^2$ and the ratio is 1.

25. In the sequence $1, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, \ldots$ the 2016th term is 2^n. Then $n =$ ———
From the pattern in the sequence given, we observe that for $k > 0$, 2^k occurs 2^k times. For example, $2^1 = 2$ occurs twice, 2^2 occurs four times and so on. The number of terms less than 2^{k+1} is

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$$

Observing that $2^{10} = 1024$ and $2^{11} = 2048$, the terms between 1024 th term and 2047 th term are all equal to 2^{10}. Thus 2016 th term is 2^{10} and $n = 10$.

26. Each root of the equation $ax^2+bx+c = 0$ is decreased by 1. The quadratic equation with these roots is $x^2+4x+1 = 0$. The numerical value of $b + c$ is ———
Let α, β be the roots of $ax^2 + bx + c = 0$. We have

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$
$$x^2 + 4x + 1 = (x - (\alpha - 1))(x - (\beta - 1))$$

Putting $x = 1$ in the first equation we get $a + b + c = a(1-\alpha)(1-\beta)$ and putting $x = 0$ in the second equation, we get $1 = (1 - \alpha)(1 - \beta)$. Thus $a + b + c = a$ and $b + c = 0$.

27. The number of integers n such that $\dfrac{n+2}{n^2+1} > \dfrac{1}{2}$ is ——

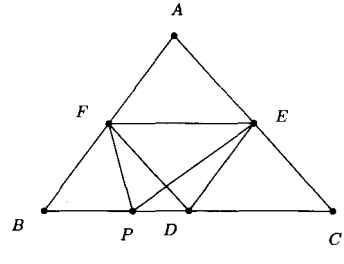
We have $2(n+2) > n^2+1$ and hence $n^2 - 2n - 3 < 0$. This gives $(n-3)(n+1) < 0$ and hence n is an integer between -1 and 3. Thus $n = 0, 1, 2$ and the number of values of n is 3.

28. P_1 and P_2 are two regular polygons. The number of sides of P_1 and P_2 respectively are in the ratio $3 : 2$ and the respective interior angles are in the ratio $10 : 9$. Then the sum of the number of sides of P_1 and P_2 is ——

The interior angle of a regular polygon of n sides is $\dfrac{(n-2)\pi}{n}$. Let P_1, P_2 have m, n sides respectively. Then $m : n = 3 : 2$ and $\dfrac{(m-2)\pi}{m} : \dfrac{(n-2)\pi}{n} = 10 : 9$. Solving for m, n we get $m = 8, n = 12$ and $m + n = 20$.

29. In triangle ABC, F and E are the mid points of AB and AC respectively. P is any point on the side BC. The ratio $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta FPE}$ is ——

If D is the mid point of the side BC, then the triangles



FPE and FDE are on the same base FE and lie between the parallel lines FE and BC. Thus their areas are equal. Since $FE = \frac{1}{2}BC, DE = \frac{1}{2}AB, DF = \frac{1}{2}AC$, it follows that area of triangle DEF is $\frac{1}{4}$ times the area of the triangle ABC. Thus $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta FPE} = 4$

30. x, y, z are distinct real numbers such that

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}$$

The value of $x^2y^2z^2$ is ———

The given equations can be written as

$$\frac{xy+1}{y} = \frac{yz+1}{z} = \frac{zx+1}{x}$$

From the first two, we obtain

$$xyz + z = y^2z + y \Rightarrow yz(x-y) = y - z$$

Similarly, we get $xy(z-x) = x-y$ and $zx(y-z) = z-x$. Multiplying these together, we get

$$x^2y^2z^2(x-y)(y-z)(z-x) = (x-y)(y-z)(z-x)$$

Since x, y, z are distinct, $(x-y)(y-z)(z-x) \neq 0$ and hence $x^2y^2z^2 = 1$.

SCREENING TEST – RAMANUJAN CONTEST
NMTC at INTER LEVEL
XI & XII Standards

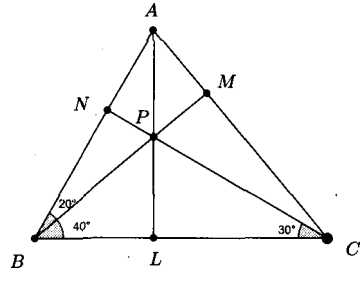
PART – A

1. P is a point on AL, the altitude of the triangle ABC through A. If $\angle PBA = 20°, \angle PBC = 40°$ and $\angle PCB = 30°$, then $\angle PCA$ equals

   A. $20°$ B. $10°$ C. $15°$

   D. $18°$

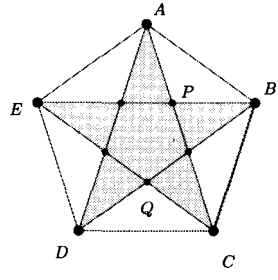
   Let BP meet AC at M and CP meet AB at N. In



   triangle BCN, $\angle BNC = 180° - 60° - 30° = 90°$. Thus CN is the altitude from C and it follows that P is the orthocenter of triangle ABC. Hence $\angle BMC = 90°$ and in triangle BNC, we have $\angle BCN = 90° - 40° = 50°$. Thus $\angle PCA = 50° - 30° = 20°$.

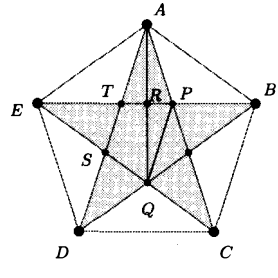
2. $ABCDE$ is a regular pentagon. The area of the star shaped region $ACEBDA$ is 1 square cm. AC and BE meet at P and BD and CE meet at Q as shown in

the figure. The area of $APQD$ in square cms is



A. $\frac{1}{4}$ B. $\frac{1}{3}$ C. $\frac{1}{2}$ D. $\frac{3}{4}$

Join AQ and let AQ. Let BE meet at R and AD and



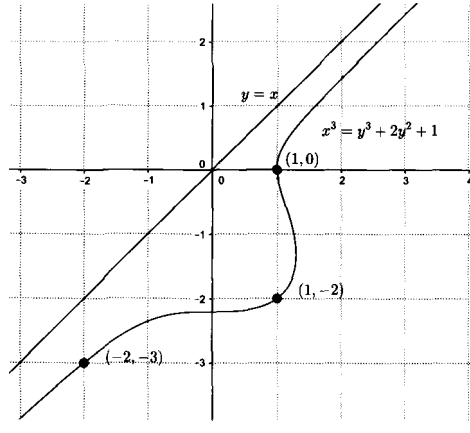
BE meet at T. The star shaped region is composed of two triangles each equal in area to AQD and two triangles each equal in area to EST. The region $APQD$ contains one triangle AQD and two triangles ARP and this equals one EST. Thus $APQD$ is composed of one AQD and one EST and thus has half the area of the star. Hence the area of $APQD$ is $\frac{1}{2}$.

3. The number of integer pairs (x, y) which satisfy the equation $x^3 = y^3 + 2y^2 + 1$ is

A. 0 B. 1 C. 2 D. 3

We have $x^3 - 1 = y^2(y + 2)$. When the left hand side is 0, we have $x = 1$ and $y = 0$ or $y = -2$. Thus we get two solutions $(1, 0)$ and $(1, -2)$. Again when $x = -2$, $x^3 - 1 = -9$ and $y^3 + 2y^2 + 9 = 0$ gives $(y+3)(y^2-y+3) = 0$ and hence $y = -3$ is a solution for this cubic. Hence we obtain another solution $(-2, -3)$. Thus we have three solutions.

The graph of the curve $x^3 = y^3 + 2y^2 + 1$ is shown below. It can be shown that there are no other integer pairs that satisfy the equation using the fact that the line $y = x$ is an asymptote to the curve.



4. Let $f(x) = x^4 + x^3 + x^2 + x + 1$. The remainder when $f(x^5)$ is divided by $f(x)$ is

A. 5 B. 6 C. 7 D. 4

For any positive integer k, $t^k - 1 = (t-1)(t^{k-1} + \cdots + 1)$. Hence putting $t = x^5$, it follows that $x^5 - 1$ divides

$x^{5k} - 1$. Now, we have

$$f(x^5) - 5 = (x^{20} + x^{15} + x^{10} + x^5 + 1) - 5$$
$$= (x^{20} - 1) + (x^{15} - 1) + (x^{10} - 1) + (x^5 - 1)$$

Each term on the right hand side is divisible by $x^5 - 1$. Since $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$, it follows that $f(x)$ divides $f(x^5) - 5$. Thus the remainder when $f(x^5)$ is divided by $f(x)$ is 5.

5. The number of positive integers a, b, c such that

$$a^2 + b^2 + c^2 = a^2b^2$$

is

A. 1 B. 2 C. 5 D. none of these

Square of an integer is either 0 mod 4 or 1 mod 4. Let us list all the possibilities mod 4:

a^2	b^2	c^2	$a^2 + b^2 + c^2$	a^2b^2	Possible solution?
0	0	0	0	0	Yes
0	0	1	1	0	No
0	1	0	1	0	No
0	1	1	2	0	No
1	0	0	1	0	No
1	0	1	2	0	No
1	1	0	2	1	No
1	1	1	3	1	No

Thus the only possibility for a solution is when all of a, b, c are even. Let $a = 2a_1, b = 2b_1, c = 2c_1$. Then we must have

$$4(a_1^2 + b_1^2 + c_1^2) = 16a_1^2b_1^2$$

and hence $a_1^2 + b_1^2 + c_1^2 = 4a_1^2b_1^2$. Arguing as above, we find that a_1, b_1, c_1 also must be even numbers. If we let $a_1 = 2a_2, b_1 = 2b_2, c_1 = 2c_2$, we must have

$$a_2^2 + b_2^2 + c_2^2 = 16a_2^2b_2^2$$

Proceeding thus, we find that a, b, c must be divisible by every power of 2 and the only possibility for this is $a = b = c = 0$. Thus the given equation has no solution in positive integers.

6. The number of triples (x, y, z) of real numbers such that $3x^2 + y^2 + z^2 = 2x(y + z)$ is

A. 1 B. 2 C. 3 D. 5

$$\begin{aligned} 3x^2 + y^2 + z^2 - 2xy - 2xz \\ &= (x^2 - 2xy + y^2) + (x^2 - 2xz + z^2) + x^2 \\ &= (x - y)^2 + (x - z)^2 + x^2 \end{aligned}$$

Thus we have $x = y = z = 0$ as the only solution.

7. The number of real roots of the equation $x^4 - 4x = 1$ is

A. 1 B. 2 C. 0 D. 4

We plot the curves $y = x^4$ and $y = 4x+1$ (See Figure 1). The intersection points are the roots of $x^4 - 4x - 1 = 0$. Clearly, there are exactly two roots for the equation. If we put $f(x) = x^4 - 4x - 1$, then $f(-1) > 0, f(0) < 0$ and $f(1) < 0$ and $f(2) > 0$. Thus one root lies between -1 and 0 and the other between 1 and 2.

8. Given a set of r points in the plane so that no three are collinear, by a closed polygon we mean the polygon obtained by connecting them by r line segments as shown in the examples below in Figure 2 (here $r = 5$).

There are 10 points on a plane no three of which are

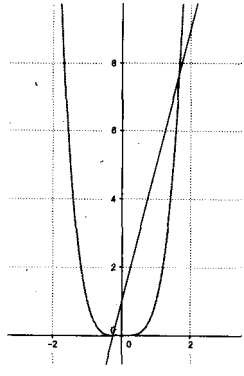


Figure 1: Figure for Q 7

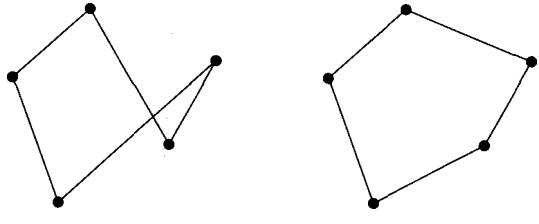
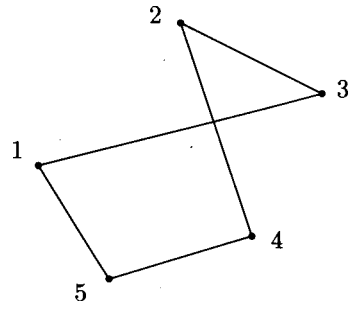


Figure 2: Figure for Q 8

collinear. The number of 5 sided closed polygons whose vertices are among these 10 points is

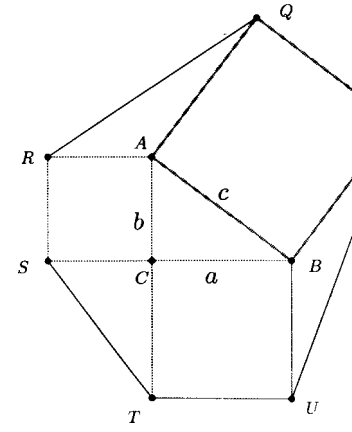
A. 6048 B. 1507 C. 3024 D. 10000

Let us first find how many 5 sided closed polygons can be formed using 5 points. Let us name the points $1, 2, 3, 4, 5$. We can connect 1 to any of the remaining 4 points, and connect that point to any of the remaining 3 points etc. and finally connect the last point to 1. Thus we can have $4 \times 3 \times 2 \times 1 = 24$ polygons. Thus any permutation of $2, 3, 4, 5$ gives a polygon. For example, the permutation 3 2 4 5 gives raise to the polygon shown



above. However, the same polygon is also the result of the permutation 5, 4, 3, 2. Thus every polygon is counted twice and hence the number of distinct polygons that could be formed with 5 points is 12. Now, we can choose 5 points from 10 points in $\binom{10}{5} = 252$ ways and for each such choice we have 12 polygons. Thus the total number of polygons with 10 points is $252 \times 12 = 3024$.

9. In the adjacent figure ABC is a right angled triangle. Squares are described externally on its sides and the outer vertices of these squares are joined as shown. If the lengths of the sides AB, BC, CA are respectively c, a, b, the area of the hexagon $PQRSTU$ is



A. $2(a^2 + ab + b^2)$

B. $a^2 + ab + b^2$

C. $3(a^2+ab+b^2)$ D. $\frac{1}{2}(a^2+ab+b^2)$

The hexagon is composed of the triangle ABC, squares $ACSR$, $ABPQ$, $CBUT$ and the triangles RAQ, SCT, UBP. Area of triangle ABC is $\frac{1}{2}ab$, of the squares mentioned above are respectively b^2, c^2, a^2. Now, in triangle PBU, $BP = c, BU = a$ and $\angle PBU = 360° - 180° - \angle B = 180° - \angle B$. Thus the area of the triangle PBU is $\frac{1}{2}ac\sin B = \frac{1}{2}ab$. Similarly, area of triangle RAQ is $\frac{1}{2}bc\sin A = \frac{1}{2}ab$ and area of triangle SCT is again $\frac{1}{2}ab$. Thus the area of the hexagon is

$$a^2+b^2+c^2+2ab = a^2+b^2+(a^2+b^2)+2ab = 2(a^2+b^2+ab)$$

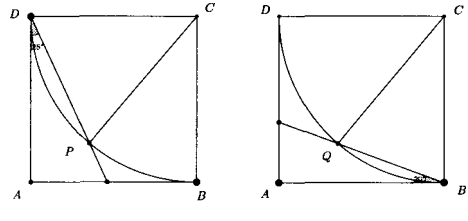
where we have used $c^2 = a^2 + b^2$

10. $ABCD$ is a square. From B, D lines are drawn to meet at P inside the square such that $\angle ADP = 25°$ and $\angle ABP = 20°$. Then $\angle BPC$ is

A. $70°$ B. $80°$ C. $60°$

D. $50°$

Let P be such that $\angle ADP = 25°$. Draw the circle



with center C and radius equal to the side of the square. This touches the side AD at D. The chord DP makes an angle $25°$ with the tangent and hence the angle subtended by this chord at the center C of the circle is $50°$. Hence $\angle PCB = 40°$.

Now suppose that the line through B making an angle $20°$ with AB meets this circle at Q. We show that Q coincides with P. Again, AB touches the circle at B and the chord BQ makes an angle $20°$ with the tangent at B. Hence the angle made by BQ at C is $40°$. Thus $\angle BCQ = \angle BCP$ and P coincides with Q. Thus $CP = CB$ and the triangle CPB is isosceles with the vertex angle $\angle BCP = 40°$. Thus $\angle BPC = 70°$.

11. If p, q are positive odd integers such that

$$(1+3+5+\cdots+p)+(1+3+5+\cdots+q) = 1+3+\cdots+19$$

then $p+q$ is

A. a prime number B. divisible by 13 C. odd number D. none of these

We have

$$1+3+\cdots+(2m-1) = m^2$$

and hence

$$1+3+5+\cdots+p = \left(\frac{p+1}{2}\right)^2$$

$$1+3+5+\cdots+q = \left(\frac{q+1}{2}\right)^2$$

$$1+3+\cdots+19 = 100$$

Thus the given equation is

$$\left(\frac{p+1}{2}\right)^2 + \left(\frac{q+1}{2}\right)^2 = 100$$

Noting that $\frac{p+1}{2}$ and $\frac{q+1}{2}$ are positive integers, the only possibility is $6^2 + 8^2 = 100$. Thus $\frac{p+1}{2} + \frac{q+1}{2} = 14$ and $p+q = 26$. Thus $p+q$ is a multiple of 13.

12. The number $2^{20} - 1$ is divisible by

A. 11 and 41 B. 11 and 21 C. 41 and 61 D. 11 and 61

We have $2^{10} - 1 = 1023 = 93 \times 11$ and hence $2^{10} = 1 \mod 11$. It follows that $2^{20} = 1 \mod 11$ and $2^{20} - 1$ is divisible by 11.

Again, $2^{10} + 1 = 1025 = 25 \times 41$. Thus $2^{10} = -1 \mod 41$. It follows that $2^{20} = 1 \mod 41$. Hence $2^{20} - 1$ is divisible by 11 and 41.

13. Five points O, A, B, C, D are taken in order on a straight line such that $OA = a$, $OB = b$, $OC = c$ and $OD = d$. P is a point on the line between B and C. If $AP : PD = BP : PC$, then OP is

A. $\dfrac{ac - bd}{a - b + c - d}$ B. $\dfrac{ac + bd}{a - b + c - d}$

C. $\dfrac{ad - bc}{a - b + c - d}$ D. none of these

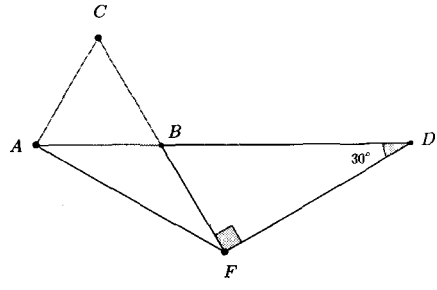
Let $OP = x$. Given that $\frac{x-a}{d-x} = \frac{x-b}{c-x}$ and hence $(x - a)(c - x) = (x - b)(d - x)$. Simplifying, we see that

$$x = \frac{ac - bd}{a - b + c - d}$$

14. The side AB of an equilateral triangle AB is produced to D such that $BD = 2AB$. The point F is the foot of the perpendicular from D on CB produced. $\angle FAC =$

A. 70° B. 75° C. 80° D. 90°

In the triangle BDF, $\angle BDF = 30°$ and hence $BF = BD \sin 30° = AB$. Thus $\angle BAF = \angle BFA$. Since $\angle BAF + \angle BFA = \angle DBF$ and $\angle DBF + \angle BFD + \angle BDF = 180°$, it follows that $\angle BAF = 30°$. Thus $\angle FAC = 90°$



15. For the simultaneous equations

$$x^2 + 2xy + y^2 - x - y = 6,$$
$$x - 2y = 3$$

A. there is a solution (x, y) such that both x, y are irrational

B. there are two sets of solutions (x, y) such that x, y are integers

C. sum of all solutions is 1

D. product of all solutions is $\frac{5}{3}$

Substituting $x = 2y + 3$ in the first equation and simplifying, we obtain $9y^2 + 15y = 0$. Thus the solutions are $y = 0, -\frac{5}{3}$. The corresponding values of x are $3, -\frac{1}{3}$. Thus the sum of all the solutions is $0 - \frac{5}{3} + 3 - \frac{1}{3} = 1$.

PART – B

16. The number of right angled triangles with integer side lengths and such that the product of the lengths of the legs (non-hypotenuse sides) equals three times the perimeter of the triangle is ———

Let s be the semi perimeter of the triangle and r its inradius. The product of the lengths of the legs is equal to twice the area of the triangle and hence equals $2rs$. Thus we are given that $2rs = 3(2s)$ and hence $r = 3$. Suppose that the right angled vertex is at the origin and the other two vertices are at $(m,0)$ and $(0,n)$ where m, n are integers. Then the line joining these two vertices $\frac{x}{m} + \frac{y}{n} = 1$ must touch the incircle $(x-3)^2 + (y-3)^2 = 9$ and hence we have

$$\left| \frac{\frac{3}{m} + \frac{3}{n} - 1}{\sqrt{\frac{1}{m^2} + \frac{1}{n^2}}} \right| = 3$$

Simplifying, we obtain

$$mn - 6m - 6n + 18 = 0$$
$$(m-6)(n-6) = 18$$

The possible values for $(m-6, n-6)$ are

$$(1,18), (2,9), (3,6), (6,3), (9,2), (18,1)$$

Hence up to congruence there are three triangles with side lengths $(7,24,25), (8,15,17)$ and $(9,12,15)$.

17. $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$x^4 - ax^3 + ax^2 + bx + c = 0$$

where a, b, c are real numbers. The smallest possible value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is ———

We have

$$\alpha + \beta + \gamma + \delta = a$$
$$\sum \alpha\beta = a$$

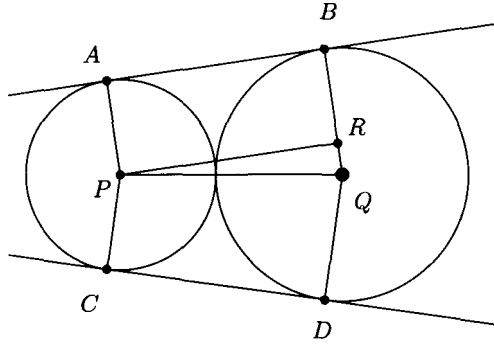
and hence

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$$
$$= a^2 - 2a = (a-1)^2 - 1$$

Since a is real, $(a-1)^2 \geq 0$ and hence the least value of $\sum \alpha^2$ is -1.

18. Two circles with centers P and Q and radii 3 and 4 respectively touch each other externally. AB, CD are direct common tangents touching the smaller circle at A, C and the bigger circle at B, D. The area of the concave hexagon $APCDQB$ is ———

Draw PR perpendicular to BQ. RQ equals the

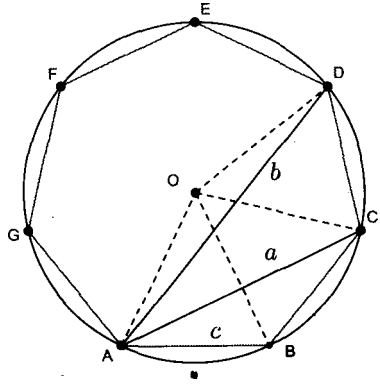


difference between the radii and hence $RQ = 1$. From the right angled triangle PRQ, $PR = \sqrt{PQ^2 - RQ^2} = \sqrt{48} = 4\sqrt{3} = AB$. The area of the trapezium $PABQ$ is $\frac{1}{2}AB(AP + BQ) = 14\sqrt{3}$. The area of the concave hexagon $APCDQB$ is twice the area of the trapezium $PABQ$ and hence equals $28\sqrt{3}$.

19. If a, b are the lengths of unequal diagonals of a regular heptagon (regular polygon with 7 sides) with side c, then

$\frac{1}{a}+\frac{1}{b}$ in terms of c is ———

Let R be the circumradius of the heptagon. The angle subtended by the diagonal a at the center of the circle is $\frac{4\pi}{7}$ and hence $a = 2R\sin\frac{2\pi}{7}$. Similarly, we have $b = 2R\sin\frac{3\pi}{7}$ and $c = 2R\sin\frac{\pi}{7}$.



$$\frac{1}{a}+\frac{1}{b} = \frac{1}{2R\sin\frac{2\pi}{7}}+\frac{1}{2R\sin\frac{3\pi}{7}} = \frac{1}{2R}\left(\frac{\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}}{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}}\right)$$

$$= \frac{1}{2R}\left(\frac{\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}}{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}}\right) = \frac{1}{2R}\left(\frac{2\sin\frac{3\pi}{7}\cos\frac{\pi}{7}}{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}}\right)$$

$$= \frac{1}{2R}\left(\frac{2\cos\frac{\pi}{7}}{\sin\frac{2\pi}{7}}\right) = \frac{1}{2R\sin\frac{\pi}{7}}$$

$$= \frac{1}{c}$$

20. The number of real roots of the equation

$$\frac{\log_{10}(\sqrt{x+1}+1)}{\log_{10}\sqrt[3]{x-40}} = 3$$

is ———

We have

$$\begin{aligned}
&\log_{10}(\sqrt{x+1}+1) = 3\log_{10}\sqrt[3]{x-40} \\
\Rightarrow\ &\log_{10}(\sqrt{x+1}+1) = \log_{10}(x-40) \\
\Rightarrow\ &\sqrt{x+1}+1 = x-40 \\
\Rightarrow\ &\sqrt{x+1} = x-41 \\
\Rightarrow\ &x+1 = (x-41)^2 \\
\Rightarrow\ &x^2-83x+1680 = 0
\end{aligned}$$

The roots are $x = 48, 35$. When $x = 48$, we have $\log_{10}(\sqrt{x+1}+1) = \log_{10}(8)$ and $3\log_{10}\sqrt[3]{x-40} = \log_{10}(8)$ and hence the equation is satisfied.
When $x = 35$, $\log_{10}(\sqrt{x+1}+1) = \log_{10}(7)$ and $3\log_{10}\sqrt[3]{x-40} = \log_{10}(-5)$ and thus $x = 35$ is not a solution. The number of real roots is 1.

21. x, y, z are non zero real numbers such that

$$x^2+y^2+z^2 = 1$$
$$x\left(\frac{1}{y}+\frac{1}{z}\right)+y\left(\frac{1}{z}+\frac{1}{x}\right)+z\left(\frac{1}{x}+\frac{1}{y}\right)+3 = 0$$

The number of possible values of $x+y+z$ is ———

From

$$x\left(\frac{1}{y}+\frac{1}{z}\right)+y\left(\frac{1}{z}+\frac{1}{x}\right)+z\left(\frac{1}{x}+\frac{1}{y}\right)+3 = 0$$

we obtain

$$x^2(y+z)+y^2(z+x)+z^2(x+y)+3xyz=0$$

We have

$$\begin{aligned}(x+y+z)^3 &= (x^2+y^2+z^2)(x+y+z)+ \\ &\quad 2x^2(y+z)+2y^2(z+x)+2z^2(x+y) \\ &\quad +6xyz \\ &= (x+y+z)\end{aligned}$$

Hence if $t=x+y+z$, we have $t^3=t$ and hence $t(t-1)(t+1)=0$. Thus $t=0,1,-1$ and hence there are three possible values for $x+y+z$.

22. The minimum value of integer n such that among any n integers we can always find three integers whose sum is divisible by 3 is ———

Consider the set $\{3,6,1,4\}$. The sum of no three integers from this set is a multiple of 3. Hence $n \geq 5$. Suppose that S is any set of n integers such that $n \geq 5$. Consider the remainders of these numbers when divided by 3. These can be 0, 1 or 2. If there are three or more numbers that leave the same remainder when divided by 3, the sum of three such numbers is divisible by 3. Suppose that each remainder occurs only a maximum of 2 times. If there are 5 or more numbers, then all the three remainders must occur in the set and hence we can choose three integers a,b,c that leave the remainders 0,1,2 respectively when divided by 3. Then clearly, $a+b+c$ is a multiple of 3. Thus $n=5$.

23. The number of integers n for which n^4-51n^2+50 is negative is ———

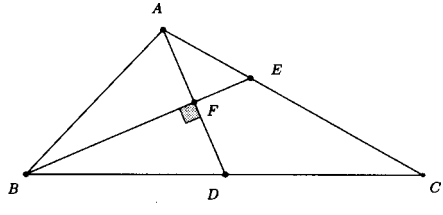
We have $n^4 - 51n^2 + 50 = (n^2 - 50)(n^2 - 1)$. This is negative when $1 < n^2 < 50$ and

$$n = \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7$$

and hence the number of such integers is 12.

24. In a triangle ABC, the lengths of the sides are consecutive integers and the median drawn from A is perpendicular to the bisector of angle B. The largest side of the triangle has length ———

It is easy to verify that the AB, BC, CA must have lengths $m-1, m+1, m$ respectively, where m is an integer. If F is the point of intersection of the median



from A and the angle bisector of $\angle B$, then triangle ABD is isosceles with $AB = BD$. Thus $2(m-1) = m+1$ and $m = 3$. Thus the largest side has length $m + 1 = 4$

25. a, b, c, d, e are real numbers such that

$$a + 4b + 9c + 16d + 25e = 1$$
$$4a + 9b + 16c + 25d + 36e = 8$$
$$9a + 16b + 25c + 36d + 49e = 23$$

The value of $a + b + c + d + e$ is ———

Adding the first and third equations and subtracting

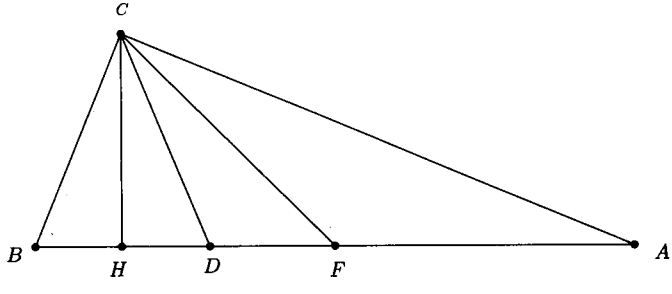
twice the second equation from the sum we get

$$2(a+b+c+d+e) = (23+1) - 16 = 8$$

and hence $a+b+c+d+e = 4$.

26. In a triangle ABC, the altitude, angle bisector and the median from C divide the angle C into four equal angles. The measure of the least angle of the triangle is ———

Let $\angle C = 4\theta$. Let CH, the altitude, CD, the angle



bisector and CE, the median divide the angle C into four equal angles. From the right angled triangle BCH, $\angle B = 90° - \theta$ and hence $\angle A = 90° - 3\theta$. Using law of sines in triangle CBF, we have

$$\frac{CF}{\cos\theta} = \frac{c/2}{\sin 3\theta}$$

and from triangle ACF,

$$\frac{CF}{\cos 3\theta} = \frac{c/2}{\sin\theta}$$

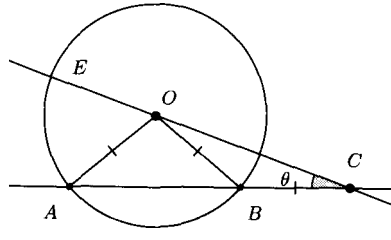
and hence

$$\frac{\cos\theta}{\sin 3\theta} = \frac{\cos 3\theta}{\sin\theta}$$

This gives $\sin 6\theta - \sin 2\theta = 0$ and $2\cos 4\theta \sin 2\theta = 0$. Since $\theta \neq 0$, it follows that $4\theta = 90°$. It follows that $\theta = 22\frac{1}{2}°$. Now, $\angle C = 90°, \angle B = 90° - \theta = 67\frac{1}{2}°$ and $\angle A = 90° - 3\theta = 22\frac{1}{2}°$. Thus the smallest angle is $22\frac{1}{2}°$

27. AB is a chord of a circle with center O. AB is produced to C such that $BC = OA$. CO is produced to E. The value of $\dfrac{\angle AOE}{\angle ACE}$ is ———

Let $\angle ACE = \theta$. Since $BC = OA = OB$, $\angle BOC = \theta$.



Thus $\angle OBA = 2\theta = \angle OAB$. Hence $\angle AOB = 180° - 4\theta$ and $\angle AOC = \angle AOB + \angle BOC = 180° - 4\theta + \theta = 180° - 3\theta$. It follows that $\angle AOE = 3\theta$ and $\frac{\angle AOE}{\angle ACE} = 3$

28. The number of two digit numbers that are less than the sum of the squares of their digits by 11 and exceed twice the product of their digits by 5 is ———

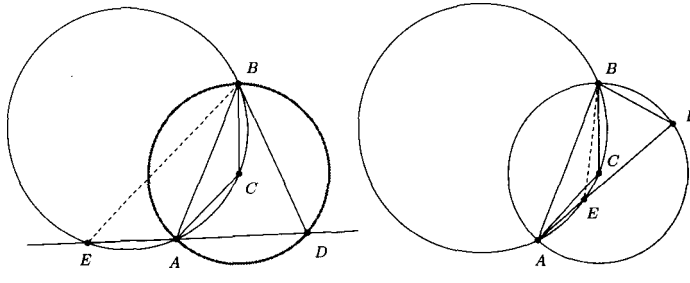
Let $N = 10a + b$ be such a number. We have $N = a^2 + b^2 - 11$ and $N = 2ab + 5$. Hence $(a - b)^2 = 16$. Thus $|a - b| = 4$ and

$$(a, b) \in \{(0,4), (4,0), (1,5), (5,1), (2,6), (6,2), (7,3), (3,7), (8,4), (4,8), (9,5), (5,9))\}$$

Of these, only $(a, b) = (1, 5), (9, 5)$ satisfies $a^2+b^2-11 = 10a + b$. Thus there are two such numbers, 15 and 95.

29. ABD is a circle whose centre is C. The circle circumscribing ABC cuts DA or DA produced at E. Then the triangle BDE is a ——— triangle.

When the circumcircle of triangle ABC cuts DA



(i) Circum circle intersects DA produced (ii) Circum circle intersects DA

Figure 3: Figure for Q29

produced at E as in (i) in the Figure 3, we have

$$\begin{aligned}
\angle BDA &= \frac{1}{2}\angle BCA = \frac{1}{2}(180^\circ - \angle BEA) \\
&= 90^\circ - \frac{1}{2}\angle BEA \\
\angle EBD &= 180^\circ - \angle BEA - \angle BDA \\
&= 180^\circ - \angle BEA - \left(90^\circ - \frac{1}{2}\angle BEA\right) \\
&= 90^\circ - \frac{1}{2}\angle BEA
\end{aligned}$$

Thus $\angle EBD = \angle EDB$ and the triangle BDE is isosceles.

When the circumcircle of triangle ABC cuts DA at E

as in (ii) in the Figure 3, we have

$$\begin{aligned}\angle BDE &= \frac{1}{2}\angle BCA = \frac{1}{2}\angle BEA \\ \angle EBD &= 180^\circ - \angle BED - \angle BDA \\ &= \angle BEA - \angle BDA \\ &= \angle BEA - \frac{1}{2}\angle BEA \\ &= \frac{1}{2}\angle BEA\end{aligned}$$

Thus $\angle BDE = \angle EBD$ and the triangle is isosceles.

30. The number of 4-digit numbers N such that

(a) no digit of N is 9

(b) N is the square of an integer

(c) when each digit of N is increased by 1, the resulting number is also square of an integer

is ———

We have $N = a^2$ and $N + 1111 = b^2$. Thus $1111 = b^2 - a^2 = (b-a)(b+a)$. Now, $1111 = 11 \times 101$ and both 11 and 101 are primes. Thus we must have $b + a = 101$ and $b - a = 11$. Thus, $b = 56$ and $a = 45$. Thus the only four digit number with the given property is $45^2 = 2025$. Answer is 1.

FINAL – GAUSS CONTEST
NMTC at PRIMARY LEVEL
V & VI Standards

1. (a) i. In how many ways can two identical balls be placed in 3 different boxes so that exactly one box is empty?

   ii. In how many ways can three identical balls be placed in 2 different boxes so that exactly one box is empty?

   iii. In how many ways can four identical balls be placed in 2 different boxes so that exactly one box is empty?

   (b) A positive integer n has five digits. N is the six digit number obtained by adjoining 2 as the leftmost digit of n. M is the six digit number by adjoining 2 at the right must digit of n. If $M = 3N$, find all the values of n.

Solution (a) (i) Let the boxes be B_1, B_2, B_3. If B_1 is empty, then the other two boxes are not empty and hence we need to place one ball in each box. Thus there is only one way in which B_1 is empty. Similarly, there is one way in which B_2 is empty and one way in which B_3 is empty. Thus there are three ways to place the balls so that exactly one box is empty.

(ii) Either we must place all the balls in the first box leaving second box empty or place all the balls in the second box leaving first box empty. Thus there are two ways in this case.

(iii) This is similar to (ii) above. Here also there are only two ways.

Solution (b) Given that

$$N = n + 200000, \qquad M = 10n + 2$$

and $10n + 2 = 3(n + 200000)$. Hence we have $7n = 599998$ and hence $n = 85714$.

2. (a) 1800 is expressed as $2^a \times 3^b \times 5^c$ and 1620 is expressed as $2^d \times 3^e \times 5^f$, where a, b, c, d, e, f are positive integers. Find the remainder when 2016 is divided by $a + b + c + d + e + f$.

   (b) Three persons A, B, C whose salaries together amount to Rs 14,400, spend 80%, 85% and 75% of their respective salaries. If their savings are as $8 : 9 : 20$, find their individual salaries.

   Solution (a) We have $1800 = 2^3 \times 3^2 \times 5^2$ and $1620 = 2^1 \times 3^4 \times 5^1$ and hence $a = 2, b = 2, c = 2, d = 1, e = 4, f = 1$. Thus $a + b + c + d + e + f = 12$ and when 2016 is divided by 12, the remainder is 0.

   (b) If their salaries are a, b, c, their savings are $0.2a$, $0.15b$, $0.25c$. We have $a + b + c = 14400$ and $0.2a : 0.15b : 0.25c = 8 : 9 : 20$. This gives $0.2a = 8k$, $0.15b = 9k$, $0.25c = 20k$ for some k. Hence $a = 40k, b = 60k, c = 80k$. Since $a + b + c = 14400$, we have $180k = 14400$ and $k = 80$. Thus $a = 40k = 3200, b = 60k = 4800, c = 80k = 6400$. Their salaries are respectively, Rs 3200, Rs 4800 and Rs 6400.

3. Completely simplify the fraction

$$\frac{7}{5 - \frac{8}{3}} \div \frac{3 - \frac{2}{3 - \frac{3}{2}}}{4 - \frac{3}{2}} - \frac{5}{7} \text{ of } \left\{ \frac{1}{1\frac{3}{7}} + \frac{6}{5} \text{ of } \frac{3\frac{1}{3} - 2\frac{1}{2}}{\frac{47}{21} - 2} \right\}$$

   By $\frac{x}{y}$ of $\frac{a}{b}$ we mean $\frac{x}{y} \times \frac{a}{b}$.

   Solution

$$\frac{7}{5 - \frac{8}{3}} \div \frac{3 - \frac{2}{3 - \frac{3}{2}}}{4 - \frac{3}{2}} = \frac{7}{7/3} \div \frac{3 - \frac{4}{3}}{5/2}$$

$$= 3 \times \frac{3}{2} = \frac{9}{2}$$

$$\frac{1}{1\frac{3}{7}} + \frac{6}{5} \text{ of } \frac{3\frac{1}{3} - 2\frac{1}{2}}{\frac{47}{21} - 2} = \frac{7}{10} + \frac{6}{5} \times \frac{10/3 - 5/2}{5/21}$$
$$= \frac{7}{10} + \frac{21}{5}$$
$$= \frac{49}{10}$$

$$\frac{5}{7} \text{ of } \left\{ \frac{1}{1\frac{3}{7}} + \frac{6}{5} \text{ of } \frac{3\frac{1}{3} - 2\frac{1}{2}}{\frac{47}{21} - 2} \right\} = \frac{5}{7} \times \frac{49}{10}$$
$$= \frac{7}{2}$$

Thus the given expression equals $\frac{9}{2} - \frac{7}{2} = 1$.

4. p, q, r are prime numbers and r is a single digit number. If $pq + r = 1993$, find $p + q + r$.
Solution r being a single digit prime number, $r = 2, 3, 5$ or 7. Note that $1993 - r$ must have only two prime factors. Since

$$1993 - 7 = 1986 = 2 \times 23 \times 41$$
$$1993 - 5 = 1988 = 4 \times 7 \times 71$$
$$1993 - 3 = 1990 = 2 \times 5 \times 199$$

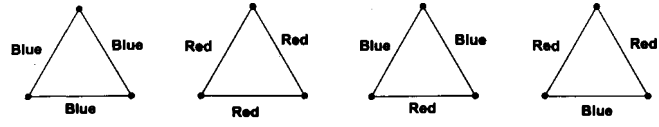
we must have $r = 2$. In this case,

$$1993 - r = 1991 = 11 \times 181$$

and thus $p + q + r = 2 + 11 + 181 = 194$.

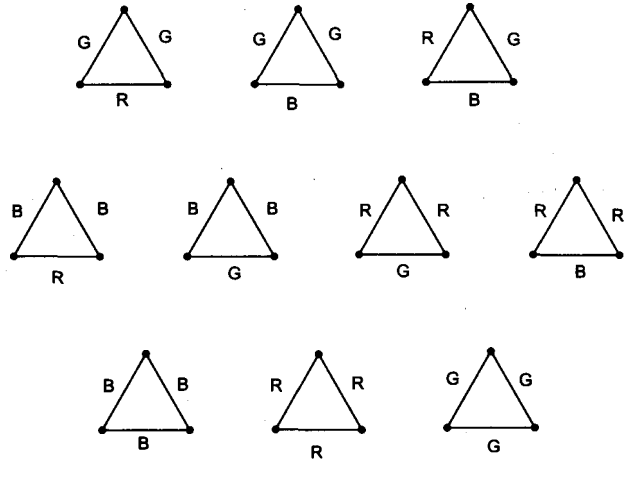
5. (a) If we have sticks of the same color and same length, we can make one triangle using them. If we have sticks of same length but two different colors, say blue and red, we can make 4 triangles as shown below.

How many triangles can be formed using sticks of

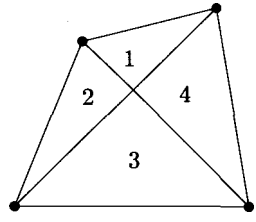


same length but three different colors, say Red, Blue and Green?

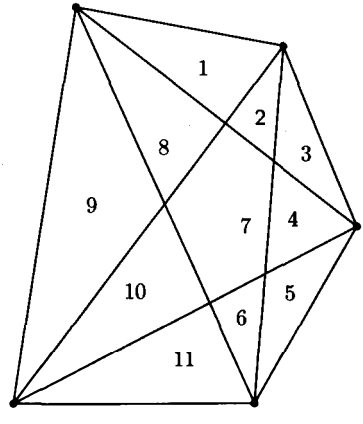
Solution (a) There are 10 triangles that can be formed with sticks of three different colors.



(b) The diagonals of a quadrilateral divide the quadrilateral into four regions. Draw a pentagon and find the maximum number of regions that can be obtained by drawing line segments connecting any two of its vertices.



Solution (b) The maximum number of regions is 11.



FINAL – KAPREKAR CONTEST
NMTC at JUNIOR LEVEL
VII & VIII Standards

1. (a) If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = 2016$, where x, y, z, a, b, c are non zero real numbers, find the value of

$$\frac{xyz(a+b)(b+c)(c+a)}{abc(x+y)(y+z)(z+x)}$$

(b) Four boys Amar, Benny, Charan, Dany and four girls Azija, Beula, Chitra and Dais have to work on a project. They should form 4 pairs, one boy and one girl in each. They know each other with the following constraints:

i. Amar knows neither Azija nor Buela

ii. Benny does not know Buela

iii. Both Charan and Dany know neither Chitra nor Daisy.

In how many ways can the pairs be formed so that each boy knows the girl in his pair.

Solution (a) We have $\frac{xyz}{abc} = (2016)^3$ and

$$2016 = \frac{x}{a} = \frac{y}{b} = \frac{x+y}{a+b}$$

and hence $\frac{a+b}{x+y} = \frac{1}{2016}$. Similarly, $\frac{b+c}{y+z} = \frac{1}{2016}$ and $\frac{c+a}{z+x} = \frac{1}{2016}$. Thus

$$\frac{xyz(a+b)(b+c)(c+a)}{abc(x+y)(y+z)(z+x)} = (2016)^3 \left(\frac{1}{2016}\right)^3 = 1$$

Solution (b) The conditions given can be tabulated as follows:

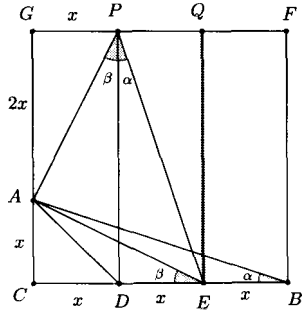
	Amar	Benny	Charan	Dany
Azija	N	Y	Y	Y
Beula	N	N	Y	Y
Chitra	Y	Y	N	N
Daisy	Y	Y	N	N

Charan and Dany can be paired only with Azija and Beula and Chitra and Daisy can only be paired with Amar and Benny. Thus there are 4 possible ways to form the pairs.

2. In a triangle ABC, $\angle C = 90°$ and $BC = 3AC$. Points D, E lie on CB such that $CD = DE = EB$. Prove that

$$\angle ABC + \angle AEC + \angle ADC = 90°$$

Solution Since $\angle ADC = 45°$, we need to show that



$\angle ABC + \angle AEC = 45°$. Complete the square $CBFG$ and let P, Q be points on GF such that $GP = PQ =$

QF. Join AP, PE. Let $CD = x$. We have

$$AP^2 = AG^2 + GP^2 = 4x^2 + x^2 = 5x^2$$
$$AE^2 = AC^2 + CE^2 = 4x^2 + x^2 = 5x^2$$

Also, from triangle PED, we have

$$PE^2 = PD^2 + DE^2 = 9x^2 + x^2 = 10x^2$$

Thus, in triangle APE, $AP = AE$ and $PE^2 = AP^2 + AE^2$ and hence it is an isosceles right angled triangle with $\angle PAE = 90°$ and $\angle APE = 45°$.
Also, noting that the triangles PDE and ACB are congruent, we have $\angle DPE = \angle ABC$ and from the congruent triangles AGP and ACE, we get $\angle GAP = \angle AEC$. Again, since CG and DP are parallel, we get $\angle GAP = \angle APD$.
Thus

$$\angle ABC + \angle AEC = \angle DPE + \angle APD = \angle APE = 45°$$

3. Let m, n, p be distinct two digit natural numbers. If

$$m = 10a + b, \quad n = 10b + c, \quad p = 10c + a$$

find all possible values of $\gcd(m, n, p)$.
Solution If d is the $\gcd(m, n, p)$ then d also divides $m + n + p = 11(a + b + c)$. If $d = 11$, then $m, n, p \in \{11, 22, \ldots, 99\}$. All these numbers are of the form $10x + x$ and hence we would have $a = b = c$, a contradiction since m, n, p are distinct. Thus $d \neq 11$.
Since d divides $a+b+c$ and the maximum value $a+b+c$ can take is $9 + 9 + 8 = 26$, we have $d \leq 26$.
Since any multiple of 5 ends with 0 or 5 and $a, b, c \neq 0$, if 5 divides d, then we must have $a = b = c = 5$, a contradiction. Thus $5 \nmid d$. Also, if $6 \mid d$, then a, b, c are all even and $a + b, b + c, c + a$ must all be multiples of 3

as well. Thus $a+b, b+c, c+a$ must be multiples of 6. Since $a, b, c \in \{2, 4, 6, 8\}$, this is impossible. Thus $6 \nmid d$. Also, if $9 \mid d$, we need $a+b, b+c, c+a$ must all be multiples of 9 and the only possibility is $(a, b, c) = (0, 0, 0)$ or $(9, 9, 9)$. Hence $9 \nmid d$.
Thus

$$d \in \{1, 2, 3, 4, 7, 8, 13, 14, 16, 17, 19, 21, 23, 26\}$$

Again, if $8 \mid d$, a, b, c are even and hence $m, n, p \in \{24, 48, 64, 88\}$. Clearly this is impossible. Again, one can easily see that $d = 17, 19, 21, 23, 26$ are impossible. Thus $d \in \{1, 2, 3, 4, 7, 13, 14\}$. For each of these cases, we give below m, n, p:

d	(m, n, p)
1	(11, 12, 21)
2	(22, 24, 42)
3	(33, 39, 93)
4	(44, 48, 84)
7	(14, 49, 91)
13	(13, 39, 91)
14	(28, 84, 42)

4. If $xy = ab(a+b)$ and

$$x^2 + y^2 - xy = a^3 + b^3$$

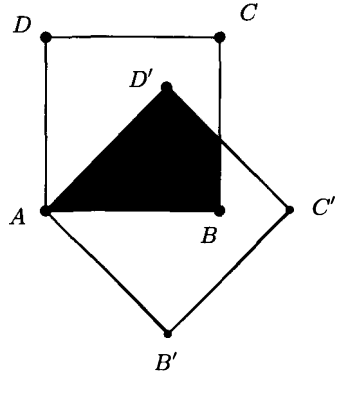
find the value of $\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{b} - \frac{y}{a}\right)$
Solution We have

$$\begin{aligned}(x+y)^2 &= x^2 + y^2 + 2xy \\ &= a^3 + b^3 + 3ab(a+b) \\ &= (a+b)^3\end{aligned}$$

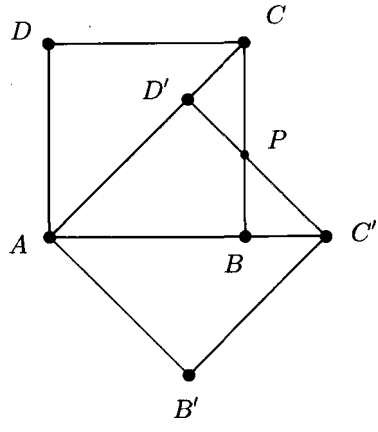
and

$$\begin{aligned}\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{b}-\frac{y}{a}\right) &= \frac{x^2+y^2}{ab} - xy\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\\ &= \frac{(x+y)^2}{ab} - \frac{2xy}{ab} - xy\left(\frac{a^2+b^2}{a^2b^2}\right)\\ &= \frac{(a+b)^3}{ab} - xy\left(\frac{(a+b)^2}{a^2b^2}\right)\\ &= \frac{(a+b)^3}{ab} - \left(\frac{xy}{ab}\right)\left(\frac{(a+b)^2}{ab}\right)\\ &= \frac{(a+b)^3}{ab} - \frac{(a+b)^3}{ab}\\ &= 0\end{aligned}$$

5. The square $ABCD$ of side length a cm is rotated about A in the clockwise direction by an angle $45°$ to become the square $AB'C'D'$. Show that the shaded area is $(\sqrt{2}-1)a^2$ square cms.



Solution Let $C'D'$ intersect BC at P. Join AC, AC'. $AD' = a$ and $AC = \sqrt{2}a$. Thus $D'C = D'P = \sqrt{2}a - a$.



Similarly, $BC' = \sqrt{2}a - a$. It follows that the triangles PCD' and $PC'B$ are congruent. Thus $D'P = PB$ and hence the required area is twice the area of triangle $AD'P$. Now, area of triangle $AD'P == \frac{1}{2}a(a\sqrt{2} - a) = (\sqrt{2} - 1)\frac{a^2}{2}$ and the required area is $(\sqrt{2} - 1)a^2$.

FINAL – BHASKARA CONTEST
NMTC at JUNIOR LEVEL
IX & X Standards

1. (a) If a, b, c are positive reals and $a + b + c = 50$ and $3a + b - c = 70$. If $x = 5a + 4b + 2c$, find the range of values of x.

   (b) The sides a, b, c of a triangle ABC satisfy the equation

$$a^2 + 2b^2 + 2016c^2 - 3ab - 4033bc + 2017ac = 0$$

   Prove that b is the arithmetic mean of a, c.

Solution (a) We have

$$(3a + b - c) - (a + b + c) = 2(a - c) = 20$$

and hence $a = c + 10 > 10$. Also

$$(3a + b - c) + (a + b + c) = 4a + 2b = 120$$

Hence $2a + b = 60$ and $a = 30 - \frac{b}{2} < 30$. Now, since $b + c = 50 - a$, we have $20 < b + c < 40$.

$$\begin{aligned} x &= 5a + 4b + 2c \\ &= (4a + 2b) + (a + b + c) + (b + c) \\ &= 120 + 50 + b + c \\ &= 170 + b + c \end{aligned}$$

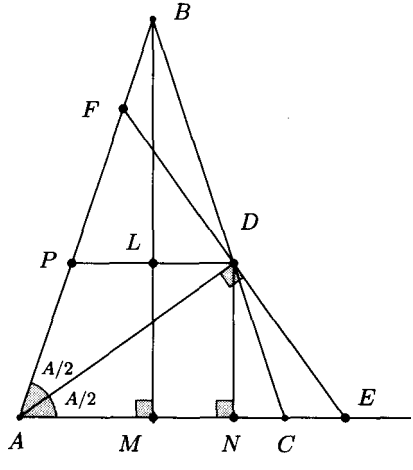
Hence $190 < x < 210$.

Solution (b) Factorizing the left hand side, the given equation can be written as

$$(a + c - 2b)(a + 2016c - b) = 0$$

Since $a + 2016c > a + c > b$, it follows that $a + c = 2b$ and thus b is the arithmetic mean of a, c.

2. In an isosceles triangle $ABC, AB = BC$. The bisector AD of $\angle A$ meets the side BC at D. The line perpendicular to AD through D meets AB at F and AC produced at E. Perpendiculars from B and D to AC are respectively BM and DN. If $AE = 2016$ units, find the length MN.
Solution Draw DP parallel to AC. Clearly,



$$\angle PDA = \angle DAC = A/2, \quad \angle PDF = \angle PFD = 90° - A/2$$

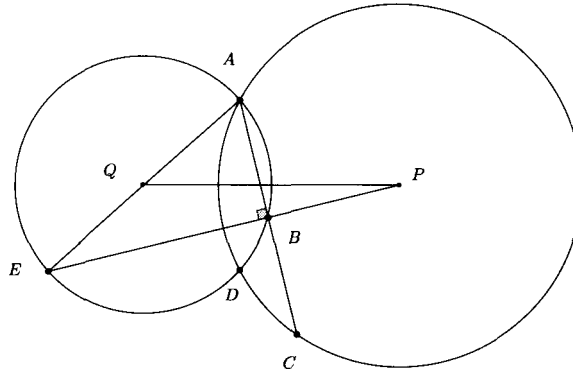
Hence P is the midpoint of AF. Let BM meet PD at L. Since ABC is isosceles, L is the mid point of PD. Thus $LD = PD/2 = MN$. But $PD = AE/2$ and thus $MN = AE/4 = 2016/4 = 504$ units.

3. (a) Two circles with centres at P and Q and radii $\sqrt{2}$ and 1 respectively intersect each other at A and D and $PQ = 2$ units. Chord AC is drawn to the bigger circle to cut it at C and the smaller circle at

B such that B is the midpoint of AC. Find the length of AC.

(b) Find the greatest common divisor of the numbers $n^n - n$, $n = 3, 5, 7, 9, \ldots$

Solution (a) Let PB produced meet the smaller circle



at E. Since B is the midpoint of AC, it follows that $PB \perp AC$ and hence AQE is a diameter of the smaller circle. Since $PQ = 2$, by Apollonius theorem, $PE^2 + PA^2 = 2(PQ^2 + QE^2)$ and hence $PE = 2\sqrt{2}$.
Let $PB = x$.
Then $BE = 2\sqrt{2} - x$ and $AB^2 = PA^2 - PB^2 = 2 - x^2$.
Also, $AB^2 = AE^2 - BE^2 = 4 - (2\sqrt{2} - x)^2$ and thus $x = \dfrac{3}{2\sqrt{2}}$. Finally, $AC = 2\sqrt{2 - \frac{9}{8}} = \sqrt{\frac{7}{2}}$

Solution (b) Since $3^3 - 3 = 24$, the GCD is at most 24.
Let $n = 2k + 1$.

$$n^n - n = n\left(n^{2k} - 1\right)$$
$$= n(n-1)(n+1)\left((n^2)^{k-1} + (n^2)^{k-2} + \cdots + 1\right)$$

One of $n-1, n, n+1$ is a multiple of 3 and $(n-1)(n+1) = 4k(k+1)$ is a multiple of 8. Hence the GCD is 24.

4. (a) A book contained problems an Algebra, Geometry and Number theory. Mahadevan solved some of them. After checking the answers, he found that he answered correctly 50% problems in Algebra, 70% in Geometry and 80% in Number theory. He further found that he solved correctly 62% of problems in Algebra and Number theory put together, 74% questions in Geometry and Number theory altogether. What is the percentage of correctly answered questions in all the three subjects?

   (b) Find all pairs of positive integers (a, b) such that $a^b - b^a = 3$.

Solution (a) Let a, g, n respectively the number of correctly answered questions in Algebra, Geometry and Number theory. Also let A, G, N be the total number of questions in Algebra, Geometry and Number theory respectively. Given that $a = 0.5A$, $g = 0.7G$, $n = 0.8N$. Also, $a + n = 0.62(A + N)$, $g + n = 0.74(G + N)$. Thus, $0.12A = 0.18N$ and $A = 1.5N$, $G = 1.5N$. Now,

$$\begin{aligned} a + g + n &= 0.5A + 0.7G + 0.8N \\ &= 0.75N + 1.05N + 0.8N = 2.6N \end{aligned}$$

Since $A + G + N = 4N$, it follows that

$$\frac{a + g + n}{A + G + N} = \frac{2.6N}{4N} = 0.65$$

Thus the percentage of correctly answered questions is 65.

Solution (b) We first prove some Lemmas that would help us solve the problem.

Lemma 1 For $n \geq 1$, $2 \leq \left(1 + \frac{1}{n}\right)^n < 3$

Proof

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+\frac{n}{1!}\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 \\ &\quad +\cdots+\frac{n(n-1)\cdots 1}{n!}\left(\frac{1}{n}\right)^n \\ &< 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} \\ &< 1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right) \\ &= 1+2=3\end{aligned}$$

Again,

$$\begin{aligned}a_n=\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} \\ &= \sum_{k=0}^{n}\frac{1}{k!}\frac{n(n-1)\cdots(n-(k-1))}{n^k} \\ &= \sum_{k=0}^{n}\frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\end{aligned}$$

For every $i \in \{1,2,\ldots,(k-1)\}$, $\left(1-\frac{i}{n}\right) < \left(1-\frac{i}{n+1}\right)$ and a_{n+1} has one more term than a_n, it follows that $a_n < a_{n+1}$. Since $a_1 = 2$, we have $2 \le a_n$ for all n.

Lemma 2 For $n \ge 1$, and $k > 1$ positive integer,

$$\left(1+\frac{k}{n}\right)^n < 3^k$$

Proof Since

$$\left(1+\frac{k}{n}\right) < \left(1+\frac{1}{n}\right)^k$$

it follows that

$$\begin{aligned}\left(1+\frac{k}{n}\right)^n &< \left(\left(1+\frac{1}{n}\right)^k\right)^n \\ &= \left(\left(1+\frac{1}{n}\right)^n\right)^k \\ &< 3^k\end{aligned}$$

Lemma 3 For any $n \geq 4$,

$$n^{n+1} - (n+1)^n > (n-1)^n - n^{n-1}$$

Proof

$$\begin{aligned}& n^{n+1} - (n+1)^n > (n-1)^n - n^{n-1} \\ \Leftrightarrow\ & n^{n+1} + n^{n-1} > (n-1)^n + (n+1)^n \\ \Leftrightarrow\ & n + \frac{1}{n} > \left(1-\frac{1}{n}\right)^n + \left(1+\frac{1}{n}\right)^n\end{aligned}$$

Since $\left(1+\frac{1}{n}\right)^n < 3$ and $\left(1-\frac{1}{n}\right)^n < 1$, we have

$$\left(1+\frac{1}{n}\right)^n + \left(1-\frac{1}{n}\right)^n < 4 < n + \frac{1}{n}$$

Lemma 4 For any $k > 1$ and $n \geq 4$,

$$n^{n+k} - (n+k)^n > n^{n+1} - (n+1)^n$$

Solution

$$\begin{aligned}& n^{n+k} - (n+k)^n > n^{n+1} - (n+1)^n \\ \Leftrightarrow\ & n^{n+k} - n^{n+1} > (n+k)^n - (n+1)^n \\ \Leftrightarrow\ & n^k - n > \left(1+\frac{k}{n}\right)^n - \left(1+\frac{1}{n}\right)^n\end{aligned}$$

Since $\left(1+\frac{k}{n}\right)^n < 3^k$ and $\left(1+\frac{1}{n}\right)^n > 2$, we have

$$\left(1+\frac{k}{n}\right)^n - \left(1+\frac{1}{n}\right)^n < 3^k - 2 < n^k - n$$

as $n > 4$.

Now suppose that $a^b - b^a = 3$. Let $a \geq 4$. If $b = a + k > a$, then we have (we have put a $\geq$ in the first line to take care of the case $k = 1$)

$$\begin{aligned} a^{a+k} - (a+k)^a &\geq a^{a+1} - (a+1)^a \\ &> (a-1)^a - a^{a-1} \\ &> \cdots \\ &> 3^4 - 4^3 = 17 \end{aligned}$$

Thus there are no solutions with $b > a \geq 4$.

If $a > b \geq 4$, then

$$a^b - b^a = -(b^a - a^b) \leq -17$$

from what we have seen above. Thus there are no solutions if either a and b are ≥ 5. Thus all solutions can be only in the range $1 \leq a, b \leq 4$. Clearly, in this range only $a = 4, b = 1$ satisfies the given equation.

5. a, b, c are positive real numbers. Find the minimum value of

$$\frac{a+3c}{a+2b+c} + \frac{4b}{a+b+2c} - \frac{8c}{a+b+3c}$$

Solution Put

$$\begin{aligned} x &= a + 2b + c \\ y &= a + b + 2c \\ z &= a + b + 3c \end{aligned}$$

Then

$$\begin{aligned} a &= -x + 5y - 3z \\ b &= x - 2y + z \\ c &= -y + z \end{aligned}$$

The given expression can be written as

$$\begin{aligned}\frac{2y-x}{x}+\frac{4(x-2y+z)}{y}+\frac{8(-y+z)}{z}&=-17+\left(2\frac{y}{x}+4\frac{x}{y}\right)+\left(4\frac{z}{y}+8\frac{y}{z}\right)\\&\geq -17+2\sqrt{8}+2\sqrt{32}\\&=-17+12\sqrt{2}\end{aligned}$$

where we have used the A.M - G.M inequality. Equality holds when

$$\frac{2y}{x}=\frac{4x}{y} \text{ and } \frac{4z}{y}=\frac{8y}{z}$$

or equivalently, when $2x = \sqrt{2}y = z$. Thus equality holds when

$$2(a+2b+c)=\sqrt{2}(a+b+2c)=(a+b+3c)$$

This can be simplified as $b=(1+\sqrt{2})a, c=(4+3\sqrt{2})a$. Thus the minimum value of the given expression is $12\sqrt{2}-17$ and is attained at

$$\left(a, b=(1+\sqrt{2})a, c=(4+3\sqrt{2})a\right)$$

with a an arbitrary positive real number.

6. (a) Show that among any $n+1$ whole numbers, one can find two numbers such that their difference is divisible by n.

   (b) Show that for any natural number n, there is a positive integer all of whose digits are 5 or 0 and is divisible by n.

Solution (a) When a number is divided by n, there are n possible remainders, $0, 1, 2, \ldots, n-1$. Since there are $n+1$ numbers, by Pigeon hole principle, some two of them, n_1, n_2, must leave the same remainder r when

divided by n and thus $n_1 = k_1 n + r$ and $n_2 = k_2 n + r$. But then $n_1 - n_2 = (k_1 - k_2)n$ and hence $n_1 - n_2$ is divisible by n.

Solution (b) Consider the numbers

$$5,\ 55,\ 555,\ \ldots \underbrace{5555\cdots555}_{n+1}$$

From the solution of (a), there are two numbers among the above n numbers whose difference is a multiple of n. If those numbers are

$$n_1 = \underbrace{5555\cdots555}_{k+l}, \qquad n_2 = \underbrace{5555\cdots555}_{l}$$

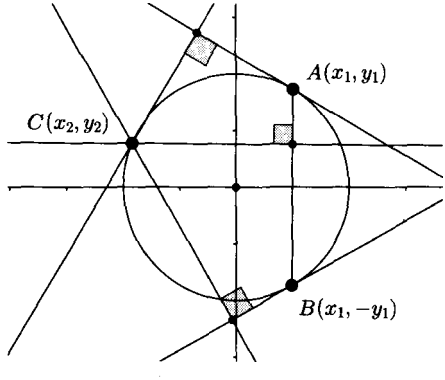
then

$$n_1 - n_2 = \underbrace{5555\cdots555}_{k}\underbrace{000\cdots0}_{l}$$

consists of 5 and 0 and is a multiple of n.

FINAL – RAMANUJAN CONTEST
NMTC at INTER LEVEL
XI & XII Standards

1. A, B, C are three points on a circle. The distance of C from the tangents at A and B to the circle are a and b respectively. If the distance of C from the chord AB is c, show that c is the geometric mean of a and b.
Solution There is no loss of generality in assuming that



the equation of the circle is $x^2+y^2 = 1$. Rotate the circle about its center so that A and B are symmetric about the X axis – that is B is the reflection of A in the X axis. If A has coordinates (x_1, y_1), the coordinates of B are $(x_1, -y_1)$. Let C be the point (x_2, y_2). The tangent at A is $xx_1 + yy_1 = 1$ and the distance of C from this tangent is

$$a = \left| \frac{x_1x_2 + y_1y_2 - 1}{\sqrt{x_1^2 + y_1^2}} \right| = |x_1x_2 + y_1y_2 - 1|$$

Similarly, the tangent at B is $xx_1 - yy_1 = 1$ and we have $b = |x_1x_2 - y_1y_2 - 1|$. Also clearly, $c = |x_2 - x_1|$.

Now,

$$\begin{aligned} ab &= |x_1x_2 + y_1y_2 - 1|\,|x_1x_2 - y_1y_2 - 1| \\ &= \left|(x_1x_2 - 1)^2 - (y_1y_2)^2\right| \\ &= \left|x_1^2x_2^2 + 1 - 2x_1x_2 - y_1^2y_2^2\right| \\ &= \left|x_1^2x_2^2 + 1 - 2x_1x_2 - (1 - x_1^2)(1 - x_2^2)\right| \\ &= \left|x_1^2 + x_2^2 - 2x_1x_2\right| \\ &= (x_1 - x_2)^2 \\ &= c^2 \end{aligned}$$

2. Find all integer solutions to the equation

$$x^3 + (x + 4)^2 = y^2$$

Solution When $x = 0$, $y^2 = 4^2$ and hence $y = \pm 4$. Thus $(x, y) = (0, \pm 4)$ are solutions. We will show there are no other solutions.
Suppose that $x \neq 0$. We have

$$x^3 = y^2 - (x + 4)^2 = (y + x + 4)(y - x - 4) \qquad (1)$$

Let d be the greatest common divisor of $y + x + 4$ and $y - x - 4$. We claim that d is a power of 2. If d has an odd prime factor $p > 1$, then clearly, p divides x^3 and hence x. Also since p divides both $y + x + 4$ and $y - x - 4$, it divides their sum, $2y$ and since p is odd, p divides y. But then p must divide 4, a contradiction.
Thus $d = 2^k$ for some $k \geq 0$.
Suppose that $k = 0$ – that is, $y + x + 4$ and $y - x - 4$ are coprime. Then, there exists integers m, n such that $y + x + 4 = m^3$ and $y - x - 4 = n^3$. We have $x = mn$ and $2x + 8 = m^3 - n^3$. Clearly, m, n are both odd and hence $|m - n| \geq 2$.

Case 1 $x = mn > 0\,,\ m > n$

$$2mn + 8 = m^3 - n^3 = (m-n)((m-n)^2 + 3mn) \geq 6mn$$

and hence $mn \leq 2$. But since m, n are both odd integers, it follows that $mn \geq 3$. Thus there are no solutions in this case.

Case 2 $x = mn > 0\,,\ m < n$

We have

$$2mn + 8 = (m-n)^3 + 3(m-n)mn \leq -8 - 6mn$$

This gives $mn \leq -2$, a contradiction to the assumption $mn > 0$.

Case 3 $x = mn < 0\,,\ m > n$

We have $2mn + 8 = m^3 - n^3 > 0$ and hence $mn > -4$. Since m, n are odd integers, the possibilities are $mn = -1, -3$. Thus $m = 1, n = -1$ or $m = 3, n = -1$. Putting $m = 1, n = -1$, we get $-1 + 27 = y^2$, and y is not an integer. For $m = 3, n = -1$, we have $-27+1 = y^2$ again an impossibility.

Case 4 $x = mn < 0\,,\ m < n$

Here, $m < 0$ and $n > 0$.

$$2mn + 8 = m^3 - n^3 < -m^2 - n^2$$

giving $m^2+n^2+2mn = (m+n)^2 < -8$, an impossibility.

Suppose that $d = 2^k$ and $k > 0$. Since 2^{2k} divides x^3, it follows that k must be a multiple of 3. We need to discuss two cases: $k > 3$ and $k = 3$. Suppose that $k > 3$. Then, $2^6 | x$. But $2^6 | (y+x+4) - (y-x-4) = 2x+8$. This implies 2^6 divides 8, a contradiction.

Suppose that $k = 3$. Then $y+x+4 = 8a, y-x-4 = 8b$ where a, b are coprime. We have $x^3 = 64ab$ and hence both a, b must be cubes of integers. Let $a = m^3, b = n^3$. Note that both m, n are odd integers. Also $x = 4mn$

and $2x + 8 = 8(m^3 - n^3)$. Thus $mn + 1 = m^3 - n^3$ and since m, n are both odd integers, $|m - n| \geq 2$. If $mn > 0$, then $m > n$ and

$$mn + 1 = (m - n)((m - n)^2 + 3mn) \geq 6mn$$

and we get $5mn \leq 1$, a contradiction. If $mn < 0$, two cases arise:

$m < 0$, $n > 0$: $2mn + 1 < mn + 1 = m^3 - n^3 \leq -m^2 - n^2$ giving $m^2 + n^2 + 2mn \leq -1$, an impossibility.
$m > 0$, $n < 0$: $mn + 1 = m^3 - n^3 > 0$ and $mn > -1$. Since $mn < 0$ and m, n are integers, this is impossible.

Thus the only solutions of the given equation are $x = 0, y = \pm 4$.

3. Two right angled triangles are such that the incircle of one triangle is equal in size to the circum circle of the other. If Δ_1 is the area of the first triangle and Δ_2, the area of the second triangle, show that $\dfrac{\Delta_1}{\Delta_2} \geq 3 + 2\sqrt{2}$

Solution Let the triangles be ABC and $A'B'C'$ as

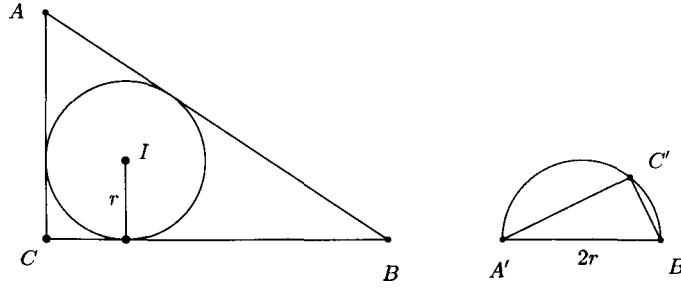


Figure 1: Triangles Δ_1 and Δ_2

shown in the Figure 1. Let the radius of the incircle of

Δ_1 be r. The radius of the circum circle of Δ_2 is also r. Since the triangles are right angled, the maximum area of Δ_2 occurs when the triangle is isosceles. Thus $\Delta_2 \leq r^2$. Now,

$$\begin{aligned}\Delta_1 &= r^2 \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\\ &= r^2 \cot\frac{A}{2}\cot\frac{B}{2}\end{aligned}$$

Now,

$$\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}} \leq \tan\left(\frac{A+B}{2}\right) = \tan\frac{\pi}{8} = \sqrt{2}-1$$

Hence

$$\begin{aligned}\Delta_1 &= r^2 \cot\frac{A}{2}\cot\frac{B}{2}\\ &\geq \frac{r^2}{(\sqrt{2}-1)^2}\\ &= r^2(\sqrt{2}+1)^2 = r^2(3+2\sqrt{2})\end{aligned}$$

and

$$\frac{\Delta_1}{\Delta_2} \geq \frac{r^2(3+2\sqrt{2})}{r^2} = 3+2\sqrt{2}$$

4. (a) Find the maximum value k for which one can choose k integers from $1, 2, \ldots, 2n$ so that none of the chosen integers is divisible by any other chosen integer.

   (b) $F(x)$ is a polynomial of degree 2016 such that all the coefficients are non negative and none exceed $F(0)$. Show that the coefficient of x^{2017} in $(F(x))^2$ is at most $\dfrac{F(1)^2}{2}$

Solution (a) If we take the set $\{n+1, n+2, \ldots, 2n\}$ then for any a, b in this set with $a < b$, a does not divide b

– since $a > n$ and $b \leq 2n$, it follows that $\frac{b}{a} < 2$ and hence $\frac{b}{a}$ can not be an integer > 1. Thus it is possible to select n numbers from the set such that no two from the set divide one another.

We will show that n is the maximum size of such a set. Suppose that we select k numbers $a_1, a_2, \ldots, a_k$. We can write $a_i = 2^{b_i}(2c_i+1)$ for some non negative integers b_i, c_i. Note that b_i is the largest power of 2 that divides a_i. Since $1 \leq a_i \neq 2n$, $2c_i+1 \in \{1, 3, \ldots, 2n-1\}$. Hence if $k \geq n+1$, we can find i, j such that $2c_i+1 = 2c_j+1$. Since $a_i = 2^{b_i}(2c_i+1)$ and $a_j = 2^{b_j}(2c_j+1)$, if $b_i \leq b_j$, then a_i divides a_j and if $b_i \geq b_j$, then a_j divides a_i. This completes the proof.

Solution (b) Let

$$F(x) = a_0 + a_1x + a_2x^2 + \cdots a_{2016}x^{2016}$$

Given that $a_i \leq a_0$ for all i.
Coefficient S of x^{2017} in $(F(x))^2$ is

$$\begin{aligned} S &= (a_1a_{2016} + a_2a_{2015} + \cdots + a_{2016}a_1) \\ &\leq a_0\,(a_1 + a_2 + \cdots + a_{2016}) \end{aligned}$$

Now

$$\begin{aligned} F(1)^2 &= (a_0 + a_1 + \cdots + a_{2016})^2 \\ &= \sum a_i^2 + 2\sum_{i<j} a_ia_j \\ &\geq 2a_0\,(a_1 + a_2 + \cdots + a_{2016}) \\ &\geq 2S \end{aligned}$$

Thus $S \leq \dfrac{F(1)^2}{2}$

5. (a) $n \geq 3$ and $a_1, a_2, \ldots, a_n$ are different positive integers. Given that, except the first and the

last, each one is a harmonic mean of its immediate neighbors. Show that none of the given integers is less than $n-1$.

(b) Show that the shortest side of a cyclic quadrilateral with circumradius 1 is at most $\sqrt{2}$.

Solution (a)

We have

$$\frac{1}{a_2}-\frac{1}{a_1}=\frac{1}{a_3}-\frac{1}{a_2}$$
$$\frac{1}{a_3}-\frac{1}{a_2}=\frac{1}{a_4}-\frac{1}{a_3}$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{1}{a_i}-\frac{1}{a_{i-1}}=\frac{1}{a_{i+1}}-\frac{1}{a_i}$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{1}{a_{n-1}}-\frac{1}{a_{n-2}}=\frac{1}{a_n}-\frac{1}{a_{n-1}}$$

and hence

$$\frac{1}{a_2}-\frac{1}{a_1}=\frac{1}{a_3}-\frac{1}{a_2}=\cdots=\frac{1}{a_{n-1}}-\frac{1}{a_{n-2}}=\frac{1}{a_n}-\frac{1}{a_{n-1}}$$

and $\frac{1}{a_1},\frac{1}{a_2},\cdots,\frac{1}{a_n}$ are in arithmetic progression. Without loss of generality, we can assume that

$$\frac{1}{a_1}>\frac{1}{a_2}>\cdots>\frac{1}{a_n}$$

Now,

$$\frac{1}{a_2}>\frac{1}{a_2}-\frac{1}{a_n}=(n-2)\left(\frac{1}{a_1}-\frac{1}{a_2}\right)$$
$$=\frac{(n-2)(a_2-a_1)}{a_1a_2}>\frac{n-2}{a_1a_2}$$

and hence $a_1>n-2$. Since a_1 is an integer, it follows that $a_1\geq n-1$. Since a_i for $i\geq 2$ are all greater that

a_1, it follows that $a_i \geq n-1$ for all i.

Solution (b)

Proof 1 If $ABCD$ is the cyclic quadrilateral, then by Ptolemy's theorem

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

If all the sides are $> \sqrt{2}$, then

$$AB \cdot CD + AD \cdot BC > 4$$

a contradiction since the largest possible length of AC, BD is 2, the length of the diameter of the circum circle.

Proof 2 Let $ABCD$ be the cyclic quadrilateral. Let $AB'C'D'$ be the square inscribed in the same circle (with vertex A' coinciding with the vertex A of the quadrilateral). Among the three vertices B, C, D if two or more are in one of the four arcs that $A'B'C'D'$ divide the circumference, then clearly the side formed by them has length at most $\sqrt{2}$. If each arc contains one vertex, then for the vertex that lies in the arc $A'B'$ or $A'D'$ the length of the side at most $\sqrt{2}$. Thus the shortest side can not have length $> \sqrt{2}$.

Proof 3 Case 1: The circum center O lies inside the quadrilateral.

Let $\alpha, \beta, \gamma, \delta$ be the angles subtended by the sides at the center of the circle. Then $\alpha + \beta + \gamma + \delta = 360°$ and hence at least one of the angles must be $\leq 90°$. The corresponding side has length at most $2\sin 45° = \sqrt{2}$. Hence the shortest side can not be longer than $\sqrt{2}$.

Case 2: The circum center O lies outside the quadrilateral.

Let α, β, γ be the angles subtended by the sides AB, BC, CD (as shown in figure) at the center of the circle. Then $\alpha + \beta + \gamma \leq 180°$ and hence at least one of

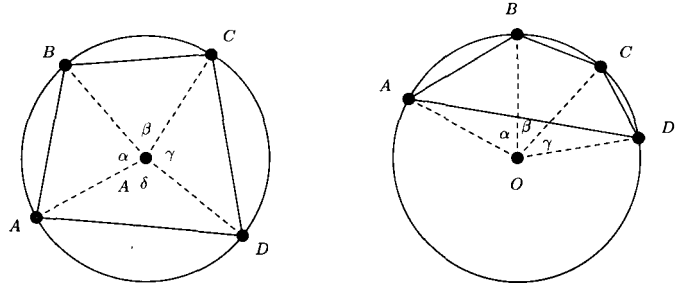


Figure 2: Cyclic Quadrilateral

the angles must be $\leq 60°$. The corresponding side has length at most $2\sin 30° = 1$. Hence, in this case, the shortest side can not be longer than 1. Clearly $1 < \sqrt{2}$.

6. C_1, C_2, C_3 are circles with radii $1, 2, 3$ respectively, touching each other as shown. Two circles can be drawn touching all these three circles. Find the radii of these two circles.

Proof 1 If A, B, C are the centers of the circles C_1, C_2, C_3 respectively, then $AB = 3, AC = 4, BC = 5$ and hence the triangle ABC is right angled. Taking A as the origin, X axis along AB and Y axis along AC, let $P(x, y)$ be the coordinates if the center of the circle that touches all the three circles externally. If r is its radius, we have $PA = r + 1, PB = r + 2, PC = r + 3$. Hence

$$x^2 + y^2 = (r+1)^2 \tag{2}$$

$$(x-3)^2 + y^2 = (r+2)^2 \tag{3}$$

$$x^2 + (y-4)^2 = (r+3)^2 \tag{4}$$

From (2) and (3) we get

$$6x - 9 = -2r - 3 \tag{5}$$

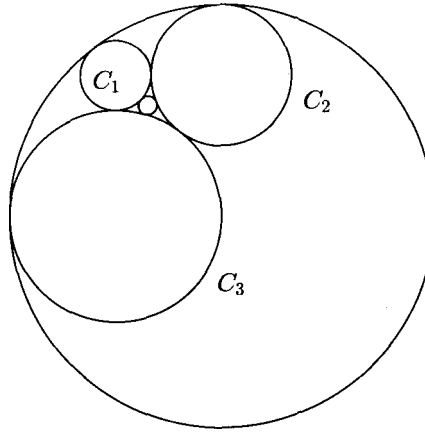


Figure 3: Circles touching each other

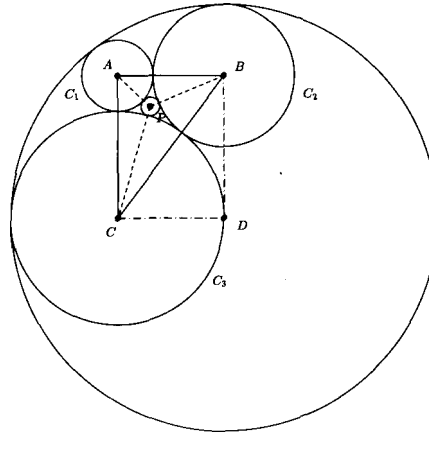


Figure 4: Circles touching each other

and from (2) and (4) we get

$$8y - 16 = -4r - 8 \tag{6}$$

Thus

$$x = \frac{6-2r}{6} = \frac{3-r}{3}, \qquad y = \frac{-4r+8}{8} = \frac{-r+2}{2}$$

Substituting these in (2), we obtain

$$\frac{9-6r+r^2}{9} + \frac{4-4r+r^2}{4} = r^2 + 2r + 1$$

and hence

$$23r^2 + 132r - 36 = 0 \Rightarrow (23r-6)(r+6) = 0$$

Thus $r = 6/23, -6$ and the radius of the smaller circle is $6/23$.

Similarly, for the circle that touches the given circles internally, we have the equations

$$x^2 + y^2 = (r-1)^2 \tag{7}$$

$$(x-3)^2 + y^2 = (r-2)^2 \tag{8}$$

$$x^2 + (y-4)^2 = (r-3)^2 \tag{9}$$

Note that these are same as (2), (3) and (4) with r replaced by $-r$ and hence these will yield

$$23r^2 - 132r - 36 = 0$$

and hence $r = 6, -6/23$. Thus the radius of the larger circle touching these is 6. This can also be seen more easily as follows: If D is the other vertex of the rectangle $ABCD$, then $DA = 5 = 6-1, DB = 4 = 6-2, DC = 3 = 6-3$ and hence the circle with center at D and radius 6 will touch all the three circles internally.

Proof 2 The circle C_4 with center D and radius $1+2+3 = 6$ touches the given circles internally. Consider the incircle of the triangle ABC. This cuts C_1, C_2, C_3 orthogonally. Hence if we invert all the circles with respect to the incircle, C_i inverts to itself and the circle

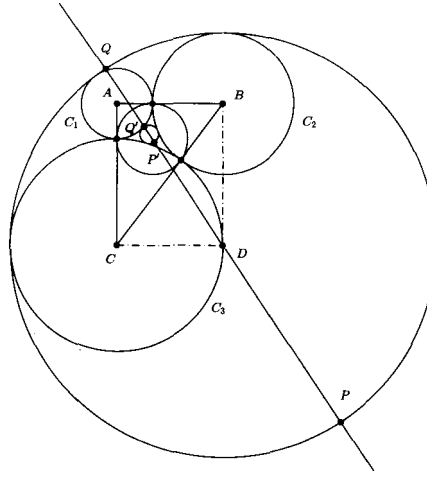


Figure 5: Proof using inversion

C_4 inverts to the circle that touches the C_i externally. To find the radius of this circle, observe that the diameter PQ of C_4 inverts to the diameter $P'Q'$ of the new circle. If I is the incenter of triangle ABC, then the coordinates of I with respect to the axes AB, AC are (r_1, r_1) and the coordinates of D are $(r_1 + r_2, r_1 + r_3)$. Thus

$$ID = \sqrt{r_2^2 + r_3^2}$$

and

$$IP = r_1 + r_2 + r_3 + \sqrt{r_2^2 + r_3^2}$$

Also

$$IQ = DQ - ID = r_1 + r_2 + r_3 - \sqrt{r_2^2 + r_3^2}$$

Thus

$$\begin{aligned} P'Q' &= IP' + IQ' \\ &= \frac{r_1^2}{IP} + \frac{r_1^2}{IQ} \\ &= \frac{r_1^2}{r_1 + r_2 + r_3 + \sqrt{r_2^2 + r_3^2}} \\ &\qquad + \frac{r_1^2}{r_1 + r_2 + r_3 - \sqrt{r_2^2 + r_3^2}} \\ &= \frac{2r_1^2(r_1 + r_2 + r_3)}{(r_1 + r_2 + r_3)^2 - (r_2^2 + r_3^2)} \\ &= \frac{2r_1^2(r_1 + r_2 + r_3)}{(r_1^2 + 2(r_1r_2 + r_2r_3 + r_3r_1)} \end{aligned}$$

Hence the radius of the smaller circle is

$$\frac{r_1^2(r_1 + r_2 + r_3)}{r_1^2 + 2(r_1r_2 + r_2r_3 + r_3r_1)}$$

In the current problem, we have $r_1 = 1, r_2 = 2, r_3 = 3$. Substituting, we get the radius as $6/23$ as before.

This also gives an easy way to construct the circles. The outer circle has center D and radius 6. The inner circle can be obtained by inversion of this circle in the incircle of triangle ABC.

Remark In the above two proofs, we used the fact that the triangle formed by the centers of the circles is right angled. In the more general case, the Descartes circle theorem helps us to find the radii of the circles that touch three mutually tangent circles. We give below a proof of Descartes' circle theorem:

Given four circles $\{C_i : 1 \leq i \leq 4\}$ that are mutually tangent to each other, their curvatures (reciprocal of radius) satisfies the following relation:

$$2(k_1^2 + k_2^2 + k_3^2 + k_4^2) = (k_1 + k_2 + k_3 + k_4)^2$$

For any three of the given four circles (say, C_2, C_3, C_4),

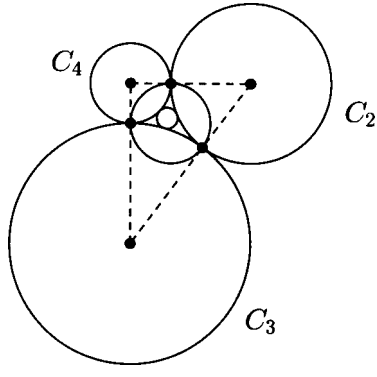


Figure 6: Circles touching externally

define a new circle that passes through their three points of contact. If the circles are externally tangent to each other as in Figure 6, then this circle is the incircle of the triangle formed by their centers. Since the sides of this triangle are $r_2+r_3, r_3+r_4, r_4+r_2$, the radius of this new circle, ρ_1 is given by

$$\rho_1 = \sqrt{\frac{r_2 r_3 r_4}{r_2 + r_3 + r_4}}$$

If we put $\kappa_1 = \rho_1^{-1}$ as its curvature and $k_i = r_i^{-1}$, the curvature of circle C_i, then we have

$$\kappa_1^2 = k_3k_4 + k_4k_2 + k_2k_3$$

If one circle, say C_2 contains other two circles as in Figure 7, then the newly defined circle is the excircle opposite to the vertex that is the center of this circle C_2. Thus

$$\rho_1 = \sqrt{\frac{r_2 r_3 r_4}{r_2 - r_3 - r_4}}$$

and again, writing the above in terms of curvatures, we have

$$\kappa_1^2 = k_3k_4 - k_4k_2 - k_2k_3$$

If we consider the curvature as negative when the circle contains the other circles, that is, putting $k_2 = -r_2^{-1}$, we can write the relation between curvatures as

$$\kappa_1^2 = k_3k_4 + k_4k_2 + k_2k_3$$

So we adopt the convention that the curvature is negative if the circle contains the other circles. Since the choices of C_2, C_3, C_4 was arbitrary, we can permute the subscripts and write three more relations between the curvatures:

$$\kappa_2^2 = k_1k_3 + k_3k_4 + k_4k_1$$
$$\kappa_3^2 = k_1k_2 + k_2k_4 + k_4k_1$$
$$\kappa_4^2 = k_1k_2 + k_2k_3 + k_3k_1$$

The four new circles are also mutually tangent to each

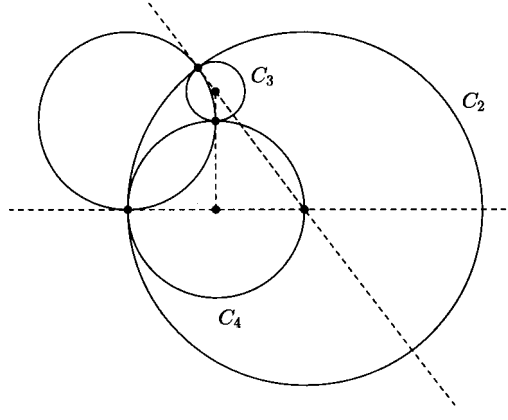


Figure 7: One circle containing others

other and this leads to the following relations:

$$k_1^2 = \kappa_3\kappa_4 + \kappa_4\kappa_2 + \kappa_2\kappa_3 \qquad k_2^2 = \kappa_1\kappa_3 + \kappa_3\kappa_4 + \kappa_4\kappa_1$$
$$k_3^2 = \kappa_1\kappa_2 + \kappa_2\kappa_4 + \kappa_4\kappa_1 \qquad k_4^2 = \kappa_1\kappa_1 + \kappa_2\kappa_3 + \kappa_3\kappa_1$$

It can be verified that the above relations hold even in the degenerate case of one of the circles becoming a straight line.

Note that there is at most one negative k_i and one negative κ_i and the corresponding circles must have the largest radii to contain the other three circles. Thus it follows that $\sum k_i > 0$ and $\sum \kappa_i > 0$.

$$\begin{aligned}\left(\sum k_i\right)^2 &= \sum k_i^2 + 2\sum_{i\neq j} k_i k_j \\ &= \sum k_i^2 + \sum \kappa_i^2 \\ &= 2\sum_{i\neq j} \kappa_i \kappa_j + \sum \kappa_i^2 \\ &= \left(\sum \kappa_i\right)^2\end{aligned}$$

Thus

$$\sum k_i = \sum \kappa_i$$

Now

$$\begin{aligned}-k_1^2 + (k_2 + k_3 + k_4)^2 &= -k_1^2 + k_2^2 + k_3^2 + k_4^2 \\ &\quad + 2k_2k_3 + 2k_3k_4 + 2k_4k_2 \\ &= -k_1^2 + k_2^2 + k_3^2 + 2\kappa_1^2 \\ &= 2\kappa_1(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) \\ &= 2\kappa_1(k_1 + k_2 + k_3 + k_4)\end{aligned}$$

Since the left hand side is

$$(k_2 + k_3 + k_4 + k_1)(k_2 + k_3 + k_4 - k_1)$$

it follows that

$$2\kappa_1 = -k_1 + k_2 + k_3 + k_4$$

Permuting the subscripts, we get

$$2\kappa_2 = k_1 - k_2 + k_3 + k_4$$
$$2\kappa_3 = k_1 + k_2 - k_3 + k_4$$
$$2\kappa_4 = k_1 + k_2 + k_3 - k_4$$

Squaring and adding these four equations, we get

$$\sum k_i^2 = \sum \kappa_i^2$$

Finally,

$$2\sum k_i^2 = \sum k_i^2 + \sum \kappa_i^2 = \left(\sum k_i\right)^2$$

When we know k_1, k_2, k_3, we can find k_4 as

$$k_4 = k_1 + k_2 + k_3 \pm 2\sqrt{k_1k_2 + k_2k_3 + k_3k_1}$$

In the present case, substituting $k_1 = 1, k_2 = \frac{1}{2}, k_3 = \frac{1}{3}$, we obtain $k_4 = -\frac{1}{6}, \frac{23}{6}$. Thus the radii of the required circles are $6, \frac{6}{23}$.

ARYABHATTA CONTEST

SENIOR – 2016

1. Suppose that the function f is defined on the half line $(0, \infty)$ by

$$f(x) = \begin{cases} 0 & \text{for } x \text{ irrational} \\ \dfrac{1}{p+q} & \text{for } x = \dfrac{p}{q},\ p,\ q \in \mathbb{N},\ \gcd(p,q) = 1 \end{cases}$$

   (a) Show that the limit of the function f exists at any point on $(0, \infty)$

   (b) At what points is the function continuous? Prove your claim.

Solution For any $a \in \mathbb{R}$, in any neighborhood of a, there are infinitely many irrationals. Hence if the limit of f exists at a, the limit must be 0.

Let $\epsilon > 0$. Choose an integer N such that $\frac{1}{N} < \epsilon$. In any interval, there are only finitely many rationals whose denominator is less than N. Thus we can choose a $\delta > 0$ such that all the rationals ($\neq a$) in $(a-\delta, a+\delta)$ contains only rationals with denominator $> N$.

Let $x \in (a-\delta, a+\delta)$ and $x \neq a$. If x is irrational, $f(x) = 0$. If x is rational, let $x = \frac{p}{q}$. Since $q > N$, we have

$$|f(x) - 0| = \left|\frac{1}{p+q}\right| < \frac{1}{N} < \epsilon$$

Thus $\lim_{x \to a} f(x) = 0$ for all x.

If a is rational, we have $\lim_{x \to a} f(x) = 0$ but $f(a) \neq 0$. Thus f is continuous only at irrationals.

2. Let C be the Cantor set which is the set of real numbers between 0 and 1 whose ternary expansion do not have

any 1. Prove that $C+C=[0,2]$ (For sets A,B, $A+B$ is defined as the set $\{a+b : a \in A, b \in B\}$).

Solution Since for all $x \in C$, we have $0 \leq x \leq 1$, it follows that $C+C \subset [0,2]$.

Consider any $x \in [0,2]$. Write $x = \frac{x}{2} + \frac{x}{2}$. Suppose that k is the first position in the ternary expansion of $\frac{x}{2}$ that is a 1. Let y be the number obtained by changing the k th position in the ternary expansion of $\frac{x}{2}$ to a 2 and z the number obtained by changing the k th position to a 0. Then $x = y + z$ and neither y nor z has 1 upto the k position in their ternary expansions. Note also that the ternary expansions of y, z are the same as that of $\frac{x}{2}$ beyond the k th position. Now find the next position in the expansion of $\frac{x}{2}$ that is a 1 and proceed as above. Finally, we obtain $y, z \in C$ such that $x = y + z$. Thus $[0,2] \subset C$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. There exists a $M > 0$ such that $|f(x) - f(y)| \geq M|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is bijective.

Solution If $x \neq y$, then $|x - y| > 0$ and hence

$$|f(x) - f(y)| \geq M|x - y| > 0$$

and hence $f(x) \neq f(y)$. Thus f is injective.

We claim that f is monotone – either strictly increasing or strictly decreasing. Suppose not. Then we can find $x_1 < x_2 < x_3$ such that $f(x_1) < f(x_2)$ and $f(x_2) > f(x_3)$ (or $f(x_1) > f(x_2)$ and $f(x_2) < f(x_3)$). Let $c \in (f(x_1), f(x_2)) \cap (f(x_3), f(x_2))$ (or $c \in (f(x_2), f(x_1)) \cap (f(x_2), f(x_3))$). By continuity, $\exists y_1 \in (x_1, x_2)$ and $y_2 \in (x_2, x_3)$ such that $f(y_1) = c = f(y_2)$. But this contradicts that f is injective. Hence f is monotonic. Suppose that f is monotonic increasing (when it is

decreasing, the proof is similar).

For $x < 0$,

$$|f(x) - f(0)| = f(0) - f(x) \geq M|x - 0| = -Mx$$

Hence $f(x) \leq f(0) + Mx$. As $x \to -\infty$, $f(x) \to -\infty$ and hence f assumes all values less than $f(0)$. Similarly, f assumes all values above $f(0)$. Hence f is surjective.

4. If α, β are two distinct roots of the equation

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^p}{p!} = 0$$

where p is a prime number > 2, prove that $\alpha+\beta, \alpha-\beta$ and $\alpha\beta$ are irrational numbers.

Solution Multiplying the equation by $p!$, α, β are also the roots of

$$f(x) = x^p + \frac{p!}{(p-1)!}x^{p-1} + \cdots + \frac{p!}{1!}x + p! = 0$$

By Eisenstein's criterion, this is irreducible and hence is the minimal polynomial of α, β over $\mathbb{Q}$, the set of rational numbers.

Suppose that $\alpha-\beta = r \in \mathbb{Q}$. Since $0 = f(\alpha) = f(\beta+r)$, β is a root of the polynomial $f(x+r)$, whose coefficients are also rational. Since $f(x)$ is the minimal polynomial of β, it follows that $f(x) \mid f(x+r)$. But both $f(x), f(x+r)$ are monic polynomials of the same degree and hence $f(x) = f(x + r)$. Thus $\alpha_1, \alpha_2, \ldots, \alpha_p$ are roots of $f(x) = 0$, then $\alpha_1 + r, \alpha_2 + r, \ldots, \alpha_p + r$ are also roots of $f(x) = 0$. Hence

$$\alpha_1 + \alpha_2 + \cdots + \alpha_p = (\alpha_1 + r) + (\alpha_2 + r) + \cdots + (\alpha_p + r)$$

Thus $pr = 0$, and $r = 0$ yielding $\alpha = \beta$, a contradiction. Thus $\alpha - \beta$ is not rational.

Suppose that $\alpha + \beta = r \in \mathbb{Q}$. Then $0 = f(\alpha) = f(r-\beta) = -f(r-\beta)$. Thus β is a root of $-f(r-x) = 0$. Since $f(x)$ is the minimal polynomial of β, it follows that $f(x) = -f(r-x)$. This gives $f\left(\frac{r}{2}\right) = -f\left(\frac{r}{2}\right)$ and hence $2f\left(\frac{r}{2}\right) = 0$. Thus $\frac{r}{2}$ is a root of $f(x) = 0$. This implies that $x - \frac{r}{2}$ divides $f(x)$, a contradiction to the irreducibility of $f(x)$. Thus $\alpha + \beta$ is not rational.

Suppose that $\alpha\beta = r \in \mathbb{Q}$. Then $0 = f(\alpha) = f\left(\frac{r}{\beta}\right)$. Thus β is a root of $\frac{x^p}{p!} f\left(\frac{r}{x}\right) = 0$ and since this is also a monic polynomial, we have $f(x) = \frac{x^p}{p!} f\left(\frac{r}{x}\right)$.

If $r > 0$, then we have $f(\sqrt{r}) = 0$. If $\sqrt{r}$ is rational, then this gives $x - \sqrt{r}$ divides $f(x)$, a contradiction. If $\sqrt{r}$ is not rational, $x^2 - r$ is the minimal polynomial of $\sqrt{r}$ and this divides $f(x)$, again a contradiction.

If $r < 0$, substituting $x = i\sqrt{-r}$ into $f(x) = \frac{x^p}{p!} f\left(\frac{r}{x}\right)$, we get $f(i\sqrt{-r}) = 0$. $x^2 + r$ is the minimal polynomial of $i\sqrt{-r}$ and it divides $f(x)$, a contradiction. Thus $\alpha\beta \notin \mathbb{Q}$.

5. Seven boxes numbered 1 to 7 are given. Seven marbles numbered 1, seven marbles numbered 2, ..., seven marbles numbered 7 are taken and placed randomly in the seven boxes such that each box has seven marbles. Now we play the following game: In stage one, a marble is drawn from box number 1 (without replacement). In stage $j \geq 2$, a marble is drawn (without replacement) from the box whose number is identical to that of the marble drawn in stage $j-1$. The game ends when one

is required to draw a marble from an empty box. Find the probability that all the marbles are drawn from all the boxes.

Solution For the game to terminate with all boxes empty, we claim that the last marble drawn must be numbered 1. Suppose not. Let a_1 be the first marble drawn from box 1 at the start of the game. If $a_1 = 1$, then six more marbles numbered 1 are drawn and after drawing the seventh marble numbered 1, the game does not terminate and hence the Box 1 must be non empty. But with a_1, seven marbles (one corresponding each of the marble numbered 1) must have been pulled out of Box 1 and still the box must be not empty, a contradiction. If $a_1 \neq 1$, then again, we have seven marbles numbered 1 are drawn and for each of these seven marbles, one marble from Box 1 is pulled out. Thus with a_1, a total of eight marbles must have been drawn from Box 1, again a contradiction. Thus if the game ends with all boxes empty, then the last marble drawn is a marble numbered 1.

Now each draw corresponds to a partial permutation of the numbered marbles. For the game to terminate with all the 49 marbles pulled out, marble 1 must be positioned in the 49th place. What about a premature termination? We extend the permutation by continuing the game with smallest numbered nonempty box and repeat the steps leading to a complete permutation of all 49 marbles. For example, suppose we have 4 boxes with the following distribution:

Box 1	Box 2	Box 3	Box 4
2134	4411	2231	3324

A premature end is achieved by drawing marbles as

follows:

2 1 1 3 1 4 2 1

We extend this permutation 21131421 by continuing the game to the permutation 2113142144324332. Again, a draw that empties all the boxes is 4211324433124321.

Thus the game terminates with all boxes empty only when the permutation ends with 1.

Is every sequence achievable in a draw of marbles? Yes. Let us see how to recover the distribution of marbles into boxes in two random sequences – one ending in 1 and the other not.

Consider the sequence 2233441123414321. Since the first marble is drawn from Box 1, put a marble numbered 2 in Box 1. The next marble drawn is 2 and must come from Box 2. So put it there. The next is 3 coming from Box 2. Thus we put the marble numbered 3 in Box 2. Proceeding similarly, we have the following distribution for this sequence:

Box 1	Box 2	Box 3	Box 4
2124	2331	3442	4113

Suppose we have the sequence 2233441123414312 not ending in 1. Proceeding as above we get the following distribution for the sequence 22334411234143

Box 1	Box 2	Box 3	Box 4
2124	233	344	4113

We have to put a 1 in the Box 3, and the last 2 in Box 1. But by now, the Box 1 is already full and so put the 2 in Box 2 which has only 3 marbles.

Box 1	Box 2	Box 3	Box 4
2124	2332	3441	4113

In this case, the game ends with the sequence 223344112341431 and the only Box remaining non empty at that point is Box 2 with a 2 in it. Since it is the smallest numbered non empty box, we proceed with drawing the marble 2 from the Box 2 to get the sequence 2233441123414312

There are $\frac{48!}{6! \times (7!)^6}$ permutations that end with 1 and the total number of permutations being $\frac{49!}{(7!)^7}$, the required probability is

$$\frac{\frac{48!}{6!\times(7!)^6}}{\frac{49!}{(7!)^7}} = \frac{1}{7}$$

6. Five people have 719 gold coins together, where each person has an integer number of coins. Further the ratio of the number of coins that a person holds with respect to the number held by those having less than him, is an integer. Find the number of coins each person has.

Solution Let the number of coins held by the different people be a, b, c, d, e. We need to find a, b, c, d, e such that $a \mid b$, $b \mid c$, $c \mid d$ and $d \mid e$ and $a+b+c+d+e = 719$. Since a divides, b, c, d, e, it divides 719. But 719 is a prime and hence $a = 1$. Thus $b + c + d + e = 718 = 2 \times 359$. Noting that 359 is a prime, we must have $b = 1$ or 2 or 359. If $b = 359$ then $c + d + e = 359$ and all

c, d, e must be multiples of 359, which is impossible. We discuss the cases $b = 1$ and $b = 2$ separately.

Case 1: $b = 1$ We have $c+d+e = 717 = 3 \times 239$. Since 239 is a prime, we have $c = 1, 3$ or 239. If $c = 239$, then $d + e = 578$ and d, e are multiples of 239, it follows that $d = e = 239$. Thus we have a solution $(a, b, c, d, e) = (1, 1, 239, 239, 239)$.

Suppose that $c = 1$. Then $d + e = 716 = 4 \times 179$ and 179 is a prime. We have the following possibilities:

$$(d, e) = (1, 715), (2, 714), (4, 712), (179, 537), (358, 358)$$

We have five solutions.

Let $c = 3$. We have $d + e = 714 = 2 \times 3 \times 7 \times 17$. Since d must be a multiple of 3, the possibilities are

$$\begin{aligned}(d, e) = {} & (3, 711), (6, 708), (21, 693), \\ & (42, 672), (51, 663), (102, 612), \\ & (357, 357)\end{aligned}$$

We have seven solutions.

This exhausts the case $b = 1$.

Case 2: $b = 2$ Here, $c + d + e = 716 = 4 \times 179$ and 179 is a prime. Since $b \mid c$, the possibilities for c are $2, 4, 358$.

When $c = 2$, $d+e = 714 = 2 \times 3 \times 7 \times 17$. Since d must be a multiple of 2, the possibilities are

$$\begin{aligned}(d, e) = {} & (2, 712), (6, 708), (14, 700), \\ & (34, 680), (42, 672), (102, 612), \\ & (238, 476)\end{aligned}$$

There are seven solutions in this case.

If $c = 4$, $d + e = 712 = 8 \times 89$. Since 89 is a prime, the

possibilities are

$$(d, e) = (4, 708),\ (8, 704),\ (356, 356)$$

Thus we obtain three more solutions.
If $c = 358$, $d + e = 358$ and d, e being multiples of 358, this gives no solutions. Thus we have a total 23 solutions listed below:

$(a, b, c, d, e) = (1, 1, 1, 1, 715), (1, 1, 1, 2, 714), (1, 1, 1, 4, 712),$
$(1, 1, 1, 179, 357), (1, 1, 1, 358, 358), (1, 1, 3, 3, 711),$
$(1, 1, 3, 6, 708), (1, 1, 3, 21, 693), (1, 1, 3, 42, 672),$
$(1, 1, 3, 51, 663), (1, 1, 3, 102, 612), (1, 1, 3, 357, 357),$
$(1, 1, 239, 239, 239), (1, 2, 2, 2, 712), (1, 2, 2, 6, 708),$
$(1, 2, 2, 14, 700), (1, 2, 2, 34, 680), (1, 2, 2, 42, 672),$
$(1, 2, 2, 102, 612), (1, 2, 2, 238, 476), (1, 2, 4, 4, 708),$
$(1, 2, 4, 8, 704), (1, 2, 4, 356, 356)$

7. Evaluate the determinant of an $n \times n$ matrix whose diagonal entries are all equal to r and off diagonal entries are all equal to s.
Solution We need to evaluate

$$D = \begin{vmatrix} r & s & s & \cdots & s \\ s & r & s & \cdots & s \\ \cdots & \cdots & \cdots & \cdots & \\ s & s & s & \cdots & r \end{vmatrix}$$

Subtract the first row from all the others to get

$$D = \begin{vmatrix} r & s & s & \cdots & s \\ s-r & r-s & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ s-r & 0 & & \cdots & r-s \end{vmatrix}$$

Taking out $r-s$ from each of the n columns, we have

$$D=(r-s)^n\begin{vmatrix} \frac{r}{r-s} & \frac{s}{r-s} & \frac{s}{r-s} & \cdots & \frac{s}{r-s} \\ -1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ -1 & 0 & & \cdots & 1 \end{vmatrix}$$

Adding all the columns to the first column, we get

$$D=(r-s)^n\begin{vmatrix} \frac{(n-1)s+r}{r-s} & \frac{s}{r-s} & \frac{s}{r-s} & \cdots & \frac{s}{r-s} \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & & \cdots & 1 \end{vmatrix}$$

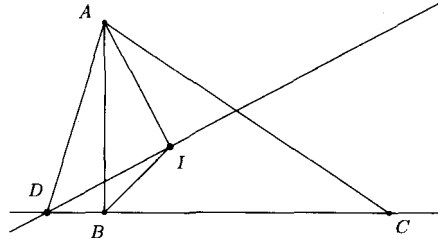
Expanding by the first column, we get

$$D=(r-s)^n\frac{(n-1)s+r}{r-s}=(r-s)^{n-1}(r+(n-1)s)$$

CRMO – 2016

1. Let ABC be a right angled triangle with $\angle B = 90°$. Let I be the incenter of ABC. Draw a line perpendicular to AI at I. Let it intersect the line CB at D. Prove that CI is perpendicular to AD and that $ID = \sqrt{b(b-a)}$, where $BC = a$ and $CA = b$.

Solution First observe that $ADBI$ is a cyclic



quadrilateral since $\angle AID = \angle ABD = 90°$. Hence $\angle ADI = \angle ABI = 45°$. Hence $\angle DAI = 45°$. But we also have

$$\begin{aligned}\angle ADB &= \angle ADI + \angle IDB \\ &= 45° + \angle IAB \\ &= \angle DAI + \angle IAC \\ &= \angle DAC\end{aligned}$$

Therefore CDA is an isosceles triangle with $CD = CA$. Since CI bisects $\angle C$, it follows that $CI \perp AD$.
This shows that $DB = CA - CB = b - a$. Therefore,

$$AD^2 = c^2 + (b-a)^2 = c^2 + b^2 + a^2 - 2ba = 2b(b-a)$$

But then $2ID^2 = AD^2 = 2b(b-a)$ and this gives $ID = \sqrt{b(b-a)}$.

2. Let a, b, c be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1$$

Prove that $abc \leq \frac{1}{8}$.

Solution We have

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c}$$
$$= \left(1 - \frac{1}{1+a}\right) + \left(1 - \frac{1}{1+b}\right) + \left(1 - \frac{1}{1+c}\right)$$

and hence

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 2$$

By $AM - GM$ inequality,

$$(1+a+1+b+1+c)\left(\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\right) \leq 9$$

and hence $a+b+c \leq \frac{3}{2}$. Now, again by $AM - GM$ inequality,

$$(abc)^{\frac{1}{3}} \leq \frac{a+b+c}{3} \leq \frac{1}{2}$$

and hence $abc \leq \frac{1}{8}$

3. For any natural number n, expressed in base 10, let $S(n)$ denote the sum of all digits of n. Find all natural numbers n such that $n = 2S(n)^2$.

Solution We use the fact that 9 divides $n - S(n)$ for every natural number n. Suppose that n is a natural number such that $n = 2S(n)^2$. We have

$$S(n)\,(2S(n) - 1) = 2S(n)^2 - S(n) = n - S(n)$$

and hence is divisible by 9. Since $S(n)$ and $2S(n) - 1$ are relatively prime, it follows that 9 divides either $S(n)$ or $2S(n) - 1$ but not both. If n has k digits, then $n \geq 10^{k-1}$ and $2S(n)^2 \leq 2 \times (9k)^2 = 162k^2$. If $k \geq 6$, we have

$$2S(n)^2 \leq 162k^2 < 5^4k^2 < 10^{k-1} \leq n$$

If $k = 5$, then

$$2S(n)^2 \leq 162 \times 25 = 4150 < 10^4 \leq n$$

Hence $k \leq 4$ and $S(n) \leq 56$.

If $9 \mid S(n)$, then $S(n) = 9, 18, 27, 36$. We see that $2S(n)^2$ is respectively equal to $162, 648, 1458, 2592$. Only 162 and 648 satisfies $n = S(n)^2$.

If $9 \mid (2S(n) - 1)$, then $2S(n) = 9k + 1$. Only $k = 1, 3, 5, 7$ give integer values for $S(n)$. In these cases, $2S(n)^2 = 50, 392, 1058, 2048$. Here again 50 and 392 give $n = 2S(n)^2$.

Thus the only natural numbers with the property $n = 2S(n)^2$ are $50, 162, 392, 648$.

4. Find the number of six digit natural numbers having exactly three odd digits and three even digits.

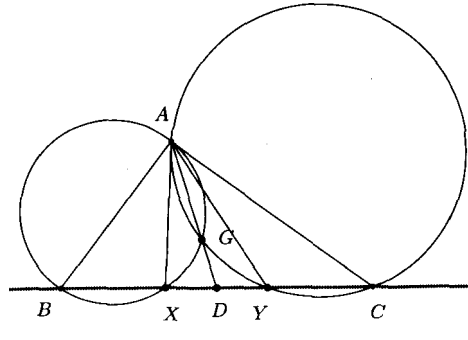
Solution First we choose 3 places for even digits. This can be done in $\binom{6}{3} = 20$ ways. Observe that the other places for odd digits get automatically fixed. There are 5 even digits and 5 odd digits. Any of these can occur in their proper places. Hence there are 5^6 ways of selecting 3 even and 3 odd digits for a particular selection of place for even digits. Hence we get 20×5^6 such numbers. But this includes all those numbers having the first digit equal to 0. Since we are looking for 6-digit numbers, these

numbers have to be removed from our counting. If we fix 0 as the first digit, we have, 2 places for even numbers and 3 places for odd numbers. We can choose 2 places for even numbers in $\binom{5}{2} = 10$ ways. As earlier, for any such choice of places for even digits, we can choose even digits in 5^2 ways and odd digits in 5^3 ways. Hence the number of ways of choosing 3 even and 3 odd digits with 0 as the first digit is 10×5^5. Therefore the number of 6-digit numbers with 3 even digits and 3 odd digits is

$$20 \times 5^6 - 10 \times 5^5 = 10 \times 5^5(10 - 1) = 281250$$

5. Let ABC be a triangle with centroid G. Let the circumcircle of the triangle AGB intersect the line BC in X different from B; and the circumcircle of the triangle AGC intersect the line BC in Y different from C. Prove that G is the centroid of the triangle AXY.

Solution Let D be the midpoint of AB. Observe that



$DX \cdot DB = DG \cdot DA = DY \cdot DC$. But $DB = DC$. Hence $DX = DY$. This means the D is the midpoint of XY as well. Hence AD is also a median of ΔAXY.

Now, we know that $AG : GD = 2 : 1$. If G' is the centroid of ΔAXY, then G' must lie on AD and $AG' : G'D = 2 : 1$. We conclude that $G = G'$.

6. Let $\langle a_1, a_2, a_3, \ldots \rangle$ be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression.
Solution Let

$$\langle a_1, a_2, a_3, \ldots \rangle = \langle a, a + d, \cdots, a + nd, \rangle$$

be a strictly increasing sequence of positive integers in arithmetic progression. Here a, d are both positive integers. Consider the following subsequence:

$$\langle a, a(1 + d), a(1 + d)^2, \ldots, a(1 + d)^n, \ldots, \rangle$$

This is a geometric progression. Here $a > 0$ and $1 + d > 1$. Hence the sequence is strictly increasing. The first term is a which is in the given AP. The second term is $a + ad$ which is the $(a + 1)$ th term of the given AP. In general,

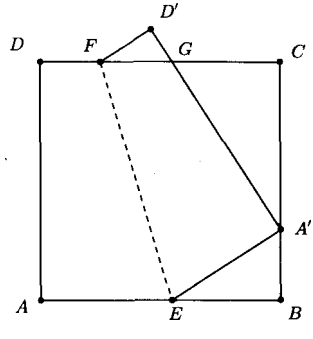
$$a(1 + d)^n = a + d\left(\binom{n}{1}a + \binom{n}{2}ad + \cdots + \binom{n}{n}ad^{n-1}\right)$$

Here the coefficient of d in the braces is also a positive integer and hence $a(1 + d)^n$ also is a term in the given AP.

INMO 2017

1. In the given figure, $ABCD$ is a square paper. It is folded along EF such that A goes to a point $A' \neq C$ on the side BC and D goes to D'. The line $A'D'$ cuts CD in G. Show that the inradius of the triangle GCA' is the sum of the inradii of the triangles $GD'F$ and $A'BE$.

Solution Observe that the triangles GCA' and $A'BE$



are similar to the triangle $GD'F$. If $GF = u$, $GD' = v$, and $D'F = w$, then we have

$$A'G = pu, \quad CG = pv, \quad A'C = pw,$$
$$A'E = qu, \quad BE = qw, \quad A'B = qv$$

If r is the inradius of $\Delta GD'F$, then pr and qr are respectively the inradii of triangles GCA' and $A'BE$. We have to show that $pr = r + qr$.
Observe that $AE = EA'$ and $DF = FD'$. We have

$$pw + qv = qw + qu = w + u + pv = v + pu$$

The last two equalities give $(p-1)(u-v) = w$. The first two equalities give $(p-q)w = q(u-v)$. Hence

$$\frac{p-q}{q} = \frac{u-v}{w} = \frac{1}{p-1}$$

Simplifying, we get $p(p-q-1)=0$. Since $p \neq 0$, we have $p=q+1$, giving $pr=qr+r$.

2. Suppose $n \geq 0$ is an integer and all the roots of $x^3+\alpha x+4-(2\times 2016^n)=0$ are integers. Find all possible values of α.
Solution Let u,v,w be the roots of the given equation. Then

$$\begin{aligned} u+v+w &= 0 \\ uv+vw+wu &= \alpha \\ uvw &= -4+2\times 2016^n \end{aligned}$$

and hence

$$uv(u+v)=4-2\times 2016^n$$

Suppose that $n \geq 1$. Then $uv(u+v)=4 \mod 2016^n$ and hence $uv(u+v)=1 \mod 3$ and $uv(u+v)=4 \mod 9$. This implies $u=2 \mod 3$ and $v=2 \mod 3$. Thus modulo 9, (u,v) could be any one of the following:

$$(2,2),\ (2,5),(2,8),(5,2),(5,5),(5,8),(8,2)(8,5),(8,8)$$

In each of the above cases, we find that $uv(u+v) \neq 4 \mod 9$. Thus $n=0$ and $uv(u+v)=2$. It follows that $(u,v)=(1,1),(1,-2)$ or $(-2,1)$. Thus

$$\alpha = uv+vw+wu = uv-(u+v)^2 = -3$$

for every pair (u,v) above.

3. Find the number of triples (x,a,b) where x is a real number and a,b belong to the set $\{1,2,3,4,5,6,7,8,9\}$ such that

$$x^2-a\{x\}+b=0$$

where $\{x\}$ denotes the fractional part of x (For example, $\{1.1\}=0.1=\{-0.9\}$).

Solution Let us write $x = n + f$, where $n = [x]$ and $f = \{x\}$. Then

$$f^2 + (2n - a)f + n^2 + b = 0 \tag{1}$$

Observe that the product of the roots of (1) is $n^2+b \geq 1$. Hence if there is a root f with $0 \leq f < 1$, the larger root must be > 1. We conclude that the equation (1) has a root < 1 only if $P(1) < 0$ where

$$P(y) = y^2 + (2n - a)y + n^2 + b$$

This gives

$$1 + (2n - a) + n^2 + b = (n + 1)^2 - a + b < 0$$

and hence $(n+1)^2+b < a$. If $n \geq 2$, then $(n+1)^2+b \geq 10 > a$. Hence $n \leq 1$. Again, if $n \leq -4$, we have $(n+1)^2 + b \geq 10 > a$. Thus $n = -3, -2, -1, 0, 1$ are the only possibilities.

If $n = -3$ or 1, then $(n+1)^2 = 4$ and we have $4+b < a$. If $a = 9$, we must have $b = 4, 3, 2, 1$ giving 4 values. For $a = 8$, we have three values for b, for $a = 7$, we have 2 values and for $a = 6$, we have 1 value. In each of these cases, we get a real value of f with $0 < f < 1$ and we obtain a solution to (1). Thus we get a total of $2(4 + 3 + 2 + 1) = 20$ values of the triple (x, a, b).

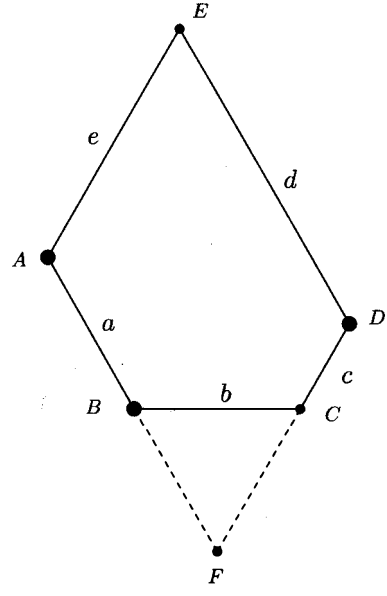
For $n = -2, 0$, we have $(n + 1)^2 = 1$. Here, we require $1 + b < a$. For $a = 9$, we have 7 values for b, for $a = 8$, we have 6 values and so on. Thus we have $2(7 + 6 + 5 + 4 + 3 + 2 + 1) = 56$ values for the triple (x, a, b).

Finally, suppose $n = -1$. Here, $(n + 1)^2 = 0$ and we need $b < a$. We get $8 + 7 + 7 + 5 + 4 + 3 + 2 + 1 = 36$ values for the triple (x, a, b).

Thus the total number of triples is $20 + 56 + 36 = 112$.

4. Let $ABCDE$ be a convex pentagon in which $\angle A = \angle B = \angle C = \angle D = 120°$ and whose side lengths are 5 consecutive integers in some order. Find all possible values of $AB + BC + CD$.

Solution Let $AB = a, BC = b, CD = c$. By symmetry,



we can assume that $c < a$. Produce DC and AB to meet at F. Then since triangle BCF is equilateral, $CF = BF = BC = b$. Also, $AF \parallel DE$ and $AE \parallel DF$. Thus in the parallelogram $AEDF$, we have $e = AE = DF = b + c$ and $d = DE = AF = a + b$.

Let the sides be $x-2, x-1, x, x+1, x+2$. Then $x \geq 3$ and $x+2 \geq (x-1)+(x-2)$ and hence $x \leq 5$. Thus $x = 3, 4$ or 5. If $x = 5$, then the sides are $\{3, 4, 5, 6, 7\}$ and

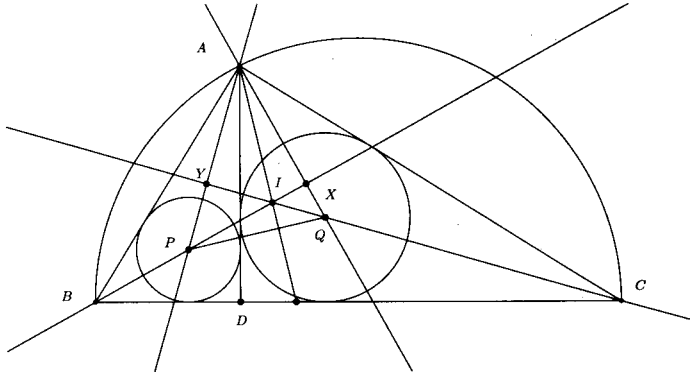
we do not have two pairs of numbers adding to another element in the set. Thus $x = 3$ or 4 and we get the sets as $\{1,2,3,4,5\}$ or $\{2,3,4,5,6\}$. With $\{1,2,3,4,5\}$, we have

$$(a,b,c,d,e) = (2,3,1,5,4) \text{ or } (4,1,2,5,3)$$

For the set $\{2,3,4,5,6\}$, we get $(a,b,c,d,e) = (4,2,3,6,5)$. Thus $a+b+c$ equals 6, 7 or 9.

5. ABC is a triangle with $\angle A = 90°$ and $AB < AC$. Let AD be the altitude from A on to BC. Let P, Q, I denote respectively the incenters of triangles ABD, ACD and ABC. Show that AI is perpendicular to PQ and $AI = PQ$.

Solution P being the incenter of the triangle ABD,



$\angle APB = 90° + \frac{1}{2}\angle ADB = 135°$. Similarly, $\angle AQC = 135°$. Thus $\angle API = \angle AQI = 45°$ (since A, P, I are collinear and C, Q, I are collinear). Also, $\angle PAQ = \frac{1}{2}\angle A = 45°$. Let $X = BI \cap AQ$ and $Y = CI \cap AP$. We have

$$\angle AXP = 180° - \angle API - \angle PAQ = 90°$$

Similarly, $\angle AYQ = 90°$. Hence I is the orthocenter of the triangle PAQ. It follows that AI is perpendicular

to PQ. Also, if R is the circum radius of the triangle PQA, then

$$AI = 2R\cos 45^\circ = 2R\sin 45^\circ = PQ$$

6. Let $n \geq 1$ be an integer and consider the sum

$$x = \sum_{k\geq 0} \binom{n}{2k} 2^{n-2k}3^k = \binom{n}{0}2^n + \binom{n}{2}2^{n-2}\cdot 3 + \cdots$$

Show that $2x-1, 2x, 2x+1$ form the sides of a triangle whose area and inradius are also integers.

Solution Consider the binomial expansion $(2+\sqrt{3})^n$. It is easy to check that

$$(2+\sqrt{3})^n = x + y\sqrt{3}$$

where x, y are also integers. Also,

$$(2-\sqrt{3})^n = x - y\sqrt{3}$$

and hence, multiplying these two equations, we get $x^2 - 3y^2 = 1$.

Since all the terms of $(2+\sqrt{3})^n$ are positive,

$$\begin{aligned} 2x &= (2+\sqrt{3})^n + (2-\sqrt{3})^n \\ &= 2\left(\binom{n}{0}2^n + \binom{n}{2}2^{n-2}\cdot 3 + \binom{n}{4}2^{n-4}\cdot 3^2 + \cdots\right) \\ &\geq 4 \end{aligned}$$

Thus $x \geq 2$ and $2x+1 < 2x + (2x-1)$ and therefore, $2x-1, 2x, 2x+1$ are sides of a triangle. The semi perimeter s is given by $2s = 2x + (2x-1) + (2x+1) = 6x$ and the area is given by

$$\Delta^2 = 3x(x+1)(x)(x-1) = 3x^2(x^2-1) = 9x^2y^2$$

Hence $\Delta = 3xy$ is an integer and the inradius is given by $\frac{\Delta}{s} = \frac{3xy}{3x} = y$ and hence is an integer.

Remark A triangle whose sides and area are all integers is called a Heronian triangle. The Indian mathematician Brahmagupta (born A.D 598) considered Heronian triangles with consecutive integer side lengths. It can be shown that all Brahmagupta triangles are of the form mentioned in this problem. The first few of them are:

n	x	Triangle
1	2	(3,4,5)
2	7	(13, 14,15)
3	26	(51, 52, 53)
4	97	(193, 194, 195)
5	362	(723, 724, 725)
6	1351	(2701, 2702, 2703)

ASSOCIATION ACTIVITIES

1. The 51st conference of the AMTI was held at Atul Vidyalaya, Gujarat on 27th, 28th and 29th December 2016. There were 197 delegates who registered of whom 189 attended. The host school, who volunteered to hold the conference, was very cooperative and helpful in looking to all logistics of the conference – boarding, lodging, entertainment and transport arrangements. All the usual programs such as endowment lectures, invited talks, panel discussions, written and oral quiz, paper presentations of teachers and students were well received. Prof. Parvin Sinclair of IGNOU delivered the invited talk on the theme and also was the moderator for the panel discussion on the theme – Mathematics through Distance Education. Some photographs of the conference are included in this issue.

2. The annual workshop for this year is scheduled from 3rd May starting with Primary to be followed by sub juniors from 10th, RMO group from 17th and teachers from 24th May 2017. About 98 students have already registered by 15th March and still continuing with 1st April as the last date. There has been an invitation from Bajaj Science centre, Wardha for Olympiad orientation for the third year in succession and three of our resource persons are planning to participate as resource persons. On invitation, the General Secretary went and gave a motivating talk to class X CBSE students at Villupuram and the response was very good besides parents demanding more frequent sessions of that type there.

3. The NMTC 49 announcement is getting ready as approved by the Executive Committee with the screening test on 26th August and final on 28th October 2017. The RMO and INMO schedules are yet to be announced as they depend on HBCSE, Mumbai. If we are to do so we plan to hold it in September 2017 separately as pre-RMO, RMO and then INMO with details to be released by 1st June 2017.

4. We have established a computer lab with 20 computers for organizing workshops and provide hands on experience to the attendees. One such for teachers of class 11 and 12 was held though with only 4 participants of which one was from Delhi. Dr. S Muralidharan was the resource person who held sessions for two days on calculus through Geogebra. On demand we may organize such workshops with about 2 to 3 week's notice for the interested who apply for the same.

5. The regular concept clarification classes on weekends, orientation for IIT etc., are being planned. At least 30 students should enlist for these to make it possible. Individuals interested can take initiative through their friends, relatives and through their children and their class mates to make these projects possible. With man power, library and other facilities we do hope and pray that aspirants will make use of these to enrich themselves well mathematically.

6. Popular lectures can be arranged in different parts of the country by interested members inviting some of the members from here or near about. Proposals for the same with details of dates, plan finances/sponsors etc. can be sent to the head quarters and after getting approval the program can be organized. The content is expected to be arousing in the recipients interest and enthusiasm towards excellence in Mathematics education.

7. The 52nd conference is scheduled at Bengaluru on 27, 28 and 29 December 2017. Please await detailed invitation in due course.

8. TIME ("Technology & Innovations in Math Education") conference is a biennial conference for Math teachers at school level. Started in 2005 by Professor Inder K Rana, this year's conference, TIME 2017 will be held during December 7 to 9 at Rajagiri School of Engineering and Technology, Kakkanad, Kerala.

Scenes from 51st conference

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27 TO 29 DEC 2016

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